

# Representation theory of Code VOA and construction of VOAs

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Dedicated to Professor Hiroshi Kimura on his 60th birthday

## Abstract

We study the representation theory of code vertex operator algebras  $M_D$  (VOAs) constructed from an even binary linear code  $D$  in [M2] and we then construct VOAs  $V$  containing a set of mutually orthogonal rational conformal vectors with central charge half such that the sum of them is the Virasoro element of  $V$  using the representation theory of  $M_D$ . The most famous example of such VOAs is the Moonshine VOA  $V^\natural$ . If a simple VOA  $V$  contains such a set of conformal vectors, then  $V$  has an elementary Abelian automorphism 2-group  $P$  generated by involutions given in [M1]. As a  $P$ -modules,  $V$  has a decomposition  $V = \bigoplus_{\chi \in Irr(P)} V^\chi$  into the direct sum of weight spaces  $V^\chi$  of  $P$ . It was proved in [DM] that  $V^\chi$  is an irreducible  $V^{1P}$ -module. Therefore, the classification of such a VOA  $V^P$  and the fusion rules of its irreducible modules will determine the structure of  $V$ . We will show that the fixed point space  $V^{1P}$  is isomorphic to a code VOA  $M_D$  of some binary linear even code  $D$  and then study and classify all irreducible  $M_D$ -modules and the fusion rules of them. Especially, a Hamming code VOA  $M_{H_8}$  of  $[8, 4, 4]$ -Hamming code  $H_8$  has a nice property that the tensor product of irreducible module with  $\mathbf{Z}/2$ -lowest weights and an irreducible module is always an irreducible module. Namely, the fusion rules of such modules have always a single component. Especially, it determines a structure of VOA containing a tensor product of Hamming code VOAs by its representations. As an application, we show a way of construction of VOA from a pair of binary codes satisfying some conditions. The key point is that vertex operators of all elements are automatically determined and the VOA structure on it is uniquely determined.

## 1 Introduction

The notion of vertex operator algebra (VOA) naturally arises from FLM's construction of the moonshine module and Borchers' insight [B], [FLM]. A VOA is essentially the chiral algebra of a two-dimensional conformal quantum field theory.

The most interesting example is the moonshine module  $V^\natural = \sum_{i=0}^{\infty} V_i^\natural$ . The construction for the moonshine module is treated in the book [FLM]. On the other hand, the simplest example is one of the minimal series of rational VOA  $L(\frac{1}{2}, 0)$  with central charge  $\frac{1}{2}$ . This is generated by one rational conformal vector with central charge  $\frac{1}{2}$ . It is known as a critical theory of two-dimensional Ising model and it has a relation with the magnetic field. Here, a rational VOA means that it has only finitely many irreducible modules and any module is a direct sum of irreducible modules.

For example,  $L(\frac{1}{2}, 0)$  has only three irreducible modules  $L(\frac{1}{2}, 0)$ ,  $L(\frac{1}{2}, \frac{1}{2})$ , and  $L(\frac{1}{2}, \frac{1}{16})$ , where the first entry is the central charge and the second denotes the lowest weights. Its fusion rules are :

$$\begin{aligned} (1) \quad & L(\frac{1}{2}, 0) \text{ is identity,} \\ (2) \quad & L(\frac{1}{2}, \frac{1}{2}) \times L(\frac{1}{2}, \frac{1}{2}) = L(\frac{1}{2}, 0), \\ (3) \quad & L(\frac{1}{2}, \frac{1}{2}) \times L(\frac{1}{2}, \frac{1}{16}) = L(\frac{1}{2}, \frac{1}{16}), \text{ and} \\ (4) \quad & L(\frac{1}{2}, \frac{1}{16}) \times L(\frac{1}{2}, \frac{1}{16}) = L(\frac{1}{2}, 0) + L(\frac{1}{2}, \frac{1}{2}). \end{aligned} \tag{1.1}$$

The remarkable property of the fusion rules for Ising model is that they give us two binary modes:

$$\bar{h} : \begin{cases} L(\frac{1}{2}, 0), L(\frac{1}{2}, \frac{1}{2}) & \rightarrow 0 \\ L(\frac{1}{2}, \frac{1}{16}) & \rightarrow 1 \end{cases} \quad \tilde{h} : \begin{cases} L(\frac{1}{2}, 0) & \rightarrow 0 \\ L(\frac{1}{2}, \frac{1}{2}) & \rightarrow 1 \end{cases}, \tag{1.2}$$

which commute with the fusion rules.

Throughout this paper, we will treat a vertex operator algebra  $V$  containing a set  $\{e^1, \dots, e^n\}$  of mutually orthogonal rational conformal vectors  $e^i$  with central charge  $\frac{1}{2}$  such that the sum  $\sum_{i=1}^n e^i$  of them is the Virasoro element  $\mathbf{w}$  of  $V$ . The most important example is the moonshine VOA  $V^\natural$ . It contains mutually orthogonal 48 conformal vectors with central charge  $\frac{1}{2}$  [DMZ]. Since each rational conformal vector  $e^i$  generates a vertex operator subalgebra  $\langle e^i \rangle \cong L(\frac{1}{2}, 0)$ ,  $V$  contains a vertex operator subalgebra  $T \cong \otimes_{i=1}^n L(\frac{1}{2}, 0)$ , that is, the tensor product of  $n$  copies of  $L(\frac{1}{2}, 0)$ . We can see  $V$  as a  $T$ -module. It is also proved in [DMZ] that every irreducible module of tensor product VOA is a tensor product of each irreducible modules. Namely, every irreducible  $T$ -module can be expressed in the form

$$\otimes_{i=1}^n L(\frac{1}{2}, h^i), \tag{1.3}$$

where  $h^i$  is one of  $0, \frac{1}{2}, \frac{1}{16}$ .

**Definition 1** We will call the above  $n$ -tuple  $(h^1, \dots, h^n)$  of each lowest weights "lowest weight row" of  $L$ . We will show that  $T$  is rational in Corollary 3.1. Let  $W$  be a  $T$ -module. Since  $\dim W_m$  is finite and the lowest weight  $\sum_{i=1}^n h^i$  of  $\otimes_{i=1}^n L(\frac{1}{2}, h^i)$  is less than

or equal to  $\frac{1}{2}n$ , we obtain that  $W$  is a finite direct sum of such irreducible  $T$ -modules. Let  $\text{lwr}(T, W)$  denote the set of all lowest weight rows of irreducible  $T$ -submodules of  $W$  with multiplicities. For each irreducible  $T$ -module  $W$  with lowest weight row  $h = (h^1, \dots, h^n)$ , we assign to it a word  $\tilde{h} = (\tilde{h}^1, \dots, \tilde{h}^n)$  by  $\tilde{h}^i = 1$  if  $h^i = \frac{1}{16}$  and  $\tilde{h}^i = 0$  otherwise. This is given by the first binary mode in (1.2). We will call this word “a  $\frac{1}{16}$ -word of  $W$ ” and denote it by  $\tilde{h}(W)$ . Even if  $W$  is not an irreducible  $T$ -module, we can use the same notation  $\tilde{h}(W)$  whenever all irreducible  $T$ -submodules of  $W$  have the same word.

It is proved by the author in [M1] that if a vertex operator algebra  $V$  contains a rational conformal vector  $e$  with central charge  $\frac{1}{2}$ , that is, if  $V$  contains  $L(\frac{1}{2}, 0)$ , then we get an automorphism  $\tau_e$  of VOA  $V$  by defining the endomorphism :

$$\tau_e : \begin{cases} 1 & \text{on } L(\frac{1}{2}, 0)\text{-submodule } U \cong L(\frac{1}{2}, 0) \text{ or } L(\frac{1}{2}, \frac{1}{2}) \\ -1 & \text{on } L(\frac{1}{2}, 0)\text{-submodule } U \cong L(\frac{1}{2}, \frac{1}{16}). \end{cases} \quad (1.4)$$

This automorphism is given by the first binary mode in (1.2). Applying to our case, we have an elementary Abelian automorphism 2-group  $P = \langle \tau_{e_i} \mid i = 1, \dots, n \rangle$  generated by  $n$  mutually commutative automorphisms. The fixed point space  $V^P$  becomes a vertex operator subalgebra of  $V$  and it coincides with the subspace generated by all irreducible  $T$ -modules whose lowest weight rows have no  $\frac{1}{16}$  entries in them. Such VOAs are constructed by the author in [M2] as code VOAs  $M_D$  from even linear codes  $D$  and we will actually show that  $V^P$  is isomorphic to  $M_D$  for some even binary linear code  $D$  in Sec.4.3 if  $V^P$  is simple. We note that if  $V$  is simple, then so is  $V^P$  by [DM]. For any linear character  $\chi$  of  $P$ , it is proved by Theorem 4.4 in [DM] that the weight space

$$V_\chi = \{v \in V \mid gv = \chi(g)v \text{ for all } g \in P\} \quad (1.5)$$

is a nontrivial irreducible  $V^P$ -module. Therefore, the classification of such code VOAs and their modules becomes important for studying these VOAs. This is our motivation of this paper.

Since every linear character  $\chi$  of  $P$  is given by a map

$$\chi : \tau_{e_i} \rightarrow (-1)^{\tilde{h}^i} \quad (\tilde{h}^i = 0, 1), \quad (1.6)$$

it assigns to  $\chi$  a binary word  $h_\chi = (\tilde{h}^1, \dots, \tilde{h}^n)$ . Therefore, the component  $V_\chi$  is generated by all irreducible  $T$ -submodules  $W$  with  $\frac{1}{16}$ -word  $\tilde{h}(W) = h_\chi$ . The set of all  $\frac{1}{16}$ -words  $h_\chi$  of  $T$ -submodules in  $V$  is closed under the sum and hence it forms a binary linear code  $S$  of length  $n$ . We will show that  $S$  is orthogonal to  $D$ .

One of the main purposes in this paper is to construct a VOA by the reverse way. Namely, assume that we are given a suitable pair of binary linear codes  $D$  and  $S$  satisfying the following conditions:

### Hypotheses II

- (1)  $D$  and  $S$  are both even linear codes.
- (2)  $\langle D, S \rangle = 0$ .
- (3) For any  $\alpha \in S$ , the weight  $|\alpha|$  is a multiple of eight.
- (4) For any  $\alpha \in S$ ,  $D$  contains a self-dual subcode  $E_\alpha$  such that  $E_\alpha$  is a direct sum  $E_\alpha = \bigoplus_{i=1}^k E_\alpha^i$  of  $[8, 4, 4]$ -Hamming codes  $E_\alpha^i$  and  $H_\alpha = \{\beta \in E_\alpha : \beta \subseteq \alpha\}$  is a direct factor of  $E_\alpha$ .
- (5) For any two codewords  $\alpha, \beta \in S$ , we assume

$$K_\alpha + E_\beta = K_{\alpha+\beta} + E_\beta, \quad (1.7)$$

where  $K_\alpha = \{\beta \in D : \beta \subseteq \alpha\}$ .

Under the hypotheses II (1)  $\sim$  (5), we will first construct a code VOA  $M_D$ . One of our assumptions is that  $D$  contains a lot of  $[8, 4, 4]$ -Hamming codes so that we can use the representation theory of Hamming code VOA  $M_{H8}$ , which has nice properties as we will see in Sec.5 and Sec.6. We will next introduce the concept of induced  $M_D$ -modules from  $M_C$ -modules for  $C \subseteq D$ . By this concept of induced module, we can define an  $M_D$ -module  $V^\alpha$  from a suitable  $M_{E_\alpha}$ -module  $U_\alpha$  for all  $\alpha \in S$  and assume that  $V^{\alpha+\beta}$  is a component of the tensor product  $V^\alpha \times V^\beta$  for any  $\alpha, \beta \in S$ . We choose  $U_\alpha$  so that  $M_{E_\alpha} \oplus U_\alpha$  has a simple VOA structure on it and assume some commutative condition for a pair  $(V^\alpha, V^\beta)$ . Set  $V = \bigoplus_{\alpha \in S} V^\alpha$ . Under these conditions, we will show that the vertex operators  $Y(v, z) \in \text{End}(V)[[z, z^{-1}]]$  for all  $v \in \bigoplus_{\alpha \in S} V^\alpha$  are uniquely determined up to change of basis and  $(V, Y)$  has a VOA structure on it.

In Sec.3, we will recall the properties of VOAs. In Sec.4, we will recall a tensor product construction of code VOA by using a vertex operator superalgebra from [M2]. In Sec.5, we will study and classify all irreducible  $M_D$ -modules. In Sec.6, we construct a VOA  $V_{H8}$  and study the fusion rules. In Sec.7, we will construct a VOA containing a tensor product  $\otimes V_{H8}$  and show that a VOA structure on it is uniquely determined up to change of basis by its representations under some conditions.

## 2 Notation

$\alpha = (a^i), \beta = (b^i), \gamma$	Words.
$\alpha^i, \beta^i$	Words in $(\mathbf{Z}_2)^8$ or $(\mathbf{Z}_2)^8/H_8$ .
$ \alpha $	The weight of binary word $\alpha = (a^i)$ , that is $ \{i \mid a^i = 1\} $ .
$D, K$	Even binary linear codes.
$e$	A conformal vector with central charge $\frac{1}{2}$ .
$\{e^i \mid i = 1, \dots, n\}$	A set of mutually orthogonal rational conformal vectors with central charge $\frac{1}{2}$ .
$\{e^i \mid i = 1, \dots, 8\}$	A set of eight coordinate conformal vectors in $V_{H_8}$ .
$E, E_\alpha$	$E = \oplus E^i$ . Direct sums of Hamming subcodes.
$\{d^i\}, \{f^i\}$	The other two sets of mutually orthogonal eight conformal vectors in $M_{H_8} = V_{H_8}$ .
$h = (h^i)$	A lowest weight row.
$\tilde{h} = (\tilde{h}^i)$	$\tilde{h}^i = \begin{cases} 1 & \text{if } h^i = \frac{1}{16} \\ 0 & \text{if } h^i = 0, \frac{1}{2} \end{cases}$ . A $\frac{1}{16}$ -word of $h$ .
$\tilde{h}(W)$	The $\frac{1}{16}$ -word of $W$ .
$H_8$	The $[8, 4, 4]$ -Hamming code.
$H(\frac{1}{2}, \alpha), H(\frac{1}{16}, \beta)$	The irreducible $V_{H_8}$ -modules, see Def.13 in Sec.6.2.
$I \begin{pmatrix} W^1 \\ W^2 & W^3 \end{pmatrix}$	The space of intertwining operators of type $\begin{pmatrix} W^1 \\ W^2 & W^3 \end{pmatrix}$ .
$\text{Ind}_{M_E}^{M_D}(U)$	The induced $M_D$ -module from an $M_E$ -module $U$ , see Sec.5.2.
$\hat{K}$	A central extension $\{\pm e^k \mid k \in K\}$ of $K$ by $\pm 1$ .
$lwr(T, W)$	The set of all lowest weight rows, see Def.1.
$L(\frac{1}{2}, 0)$	The rational Virasoro VOA with central charge $\frac{1}{2}$ .
$L(\frac{1}{2}, h)$	An irreducible $L(\frac{1}{2}, 0)$ -module with lowest weight $h$ .
$M$	A minimal SVOA $M = M_{\bar{0}} \oplus M_{\bar{1}}$ , see Sec.4.1.
$M_{\bar{0}}, M_{\bar{1}}$	The even part and the odd part of SVOA $M$ .
$M_\alpha$	$= (\otimes_{\alpha=(a^i)} (M^i)_{\overline{a^i}}) \otimes e^d$ .
$M_S$	$= \sum_{\alpha \in S} M_\alpha$ .
$\hat{M}_\alpha$	$= \otimes_{\alpha=(a^i)} (M^i)_{\overline{a^i}}$ .
$\hat{M}_S$	$= \sum_{\alpha \in S} \hat{M}_\alpha$ .
$M_D$	A VOA constructed from an even binary linear code $D$ .

$N$	A sufficiently large integer.
$\mathbf{N}$	$= \{0, 1, 2, \dots\}$ . The set of nonnegative integers.
$\nu_i$	$\nu_i = (0^{s-1}10^{n-i}) = \{i\}$ .
$P$	$= \langle \tau_{e_i} \mid i = 1, \dots, n \rangle$ .
$q$	A lowest weight vector $q = \frac{1}{\sqrt{2}}(e^x + e^{-x}) \in M_{\bar{1}}$ , see (4.9).
$q^i$	A copy of $q$ in $M^i$ , see (4.33).
$q^{a^i}$	$\begin{cases} = \mathbf{1}^i \in M_0^i & \text{if } a^i = 0 \\ = q^i \in M_0^i & \text{if } a^i = 1. \end{cases}$
$q^\alpha$	$\otimes q^{a^i}$ . See (4.33).
$Q$	A $\hat{K}$ -module.
$\mathbf{Q}$	The field of rational numbers.
$\{s^\alpha : \alpha \in \mathbf{Z}_2^8/H_8\}$	The other sixteen conformal vectors in $V_{H_8}$ , see (6.3).
$\sigma_e$	An automorphism of VOA given by $e$ of type 2, see [M1].
Support of $D$	$= \{i \mid \exists (a^i) \in D \text{ s.t. } a^i \neq 0\}$
$T$	$= \otimes_{i=1}^n L(\frac{1}{2}, 0)$ , see Sec.1.
$\tau_e$	An automorphism of VOA given by $e$ , see [M1].
$\times$	A fusion rule or a tensor product.
$V_{H_8}$	$= M_{H_8}$ . The VOA constructed by $H_8$ , see Sec.6.
$\mathbf{w}, \mathbf{w}^i$	Virasoro elements.
$Y(v, z)$	$= \sum_{n \in \mathbf{Q}} v_n z^{-n-1}$ . A vertex operator of $v$ .
$\mathbf{Z}$	The ring of integers.
$\mathbf{Z}_2$	$= \{0, 1\}$ . The binary field.
$A(x, y) \sim B(x, y)$	$\Leftrightarrow \exists N \in \mathbf{N} \text{ s.t. } (x - y)^N (A(x, y) - B(x, y)) = 0$
$(1^m 0^n)$	$= (1 \dots 10 \dots 0)$
$\mathbf{1}, \mathbf{1}^i$	The vacuum.

## 3 Vertex operators

### 3.1 Properties of VOAs

Throughout this paper we will treat a VOA  $V = \oplus_{i=0}^\infty V_n$  such that  $\dim V_0 = 1$ . First, let us recall some basic properties of VOAs.

**Definition 2** *Let  $V$  be a vector space. We will use the following notations:*

$$R[[z, z^{-1}]] = \{\sum_{n \in \mathbf{Z}} a_n z^{-n-1} : a_n \in R\},$$

$$R\{z, z^{-1}\} = \{\sum_{n \in \mathbf{C}} a_n z^{-n-1} : a_n \in R\}.$$

A weak vertex operator of  $V$  is a formal power series

$v(z) = \sum_{n \in \mathbf{Z}} v_n z^{-n-1} \in (\text{End} V)[[z, z^{-1}]]$  so that for any  $u \in V$  there is an integer  $N(u) > 0$  such that  $v_n u = 0$  for  $n > N(u)$ .

**Lemma 3.1 (Li, Dong)** Let  $V = \sum_{n=0}^{\infty} V_n$  be a Fock space and let  $u(z) = \sum u_n z^{-n-1}$ ,  $v(z) = \sum v(m) z^{-m-1}$  be weak vertex operators of  $V$ . For any integer  $n$ , we define the  $n$ -th normal product by

$$u(z)_n v(z) = \text{Res}_{z_1} ((z_1 - z)^n u(z_1) v(z) - (-z + z_1)^n v(z) u(z_1)). \quad (3.1)$$

Then  $u(z)_n v(z)$  is also a weak vertex operator of  $V$ . Here, the binomial coefficients, the binomial expansions and the residue  $\mathbf{Res}_{z_1}$  are defined by

$$\begin{aligned} \binom{n}{i} &= \frac{n(n-1) \cdots (n-i+1)}{i(i-1) \cdots 1}, \\ (z_1 + z)^n &= \sum_{i=0}^{\infty} \binom{n}{i} z_1^{n-i} z^i, \\ \mathbf{Res}_{z_1} (\sum a_n z_1^{-n-1}) &= a_0. \end{aligned}$$

We note that  $(z_1 + z)^n \neq (z + z_1)^n$  in our notation if  $n$  is not a natural number. Originally,  $(z_1 + z)^n$  and  $(z + z_1)^n$  were written by  $l_{z_1, z}(z_1 + z)^n$  and  $l_{z, z_1}(z_1 + z)^n$ .

The next Dong's lemma is important, (see [L1].)

**Lemma 3.2 (Dong)** Let  $u(z), v(z), w(z)$  be weak vertex operators of  $V$ . Assume that  $u(z), v(z)$  and  $w(z)$  satisfy the mutual commutativity, then for any integer  $n$ ,  $u(z)_n v(z)$  satisfy the commutativity with  $w(z)$ . Here the commutativity means

$$(z_1 - z_2)^N u(z_1) v(z_2) = (z_1 - z_2)^N v(z_2) u(z_1) \quad (3.2)$$

for a sufficient large integer  $N$ .

It follows from an easy calculation that the normal product also preserves the derivation.

**Lemma 3.3** Let  $u(z), v(z)$  be weak vertex operators of  $V$  and  $L(-1) \in \text{End}(V)$ . If  $[L(-1), u(z)] = \frac{d}{dz} u(z)$  and  $[L(-1), v(z)] = \frac{d}{dz} v(z)$ , then  $[L(-1), u(z)_n v(z)] = \frac{d}{dz} (u(z)_n v(z))$ .

We will write only the essential parts of the axioms of VOA, its modules, and intertwining operators here. See [FLM] or the others for the detail.

**Definition 3** *A vertex operator algebra is a  $\mathbf{Z}$ -graded vector space  $V = \sum_{n=0}^{\infty} V_n$  with finite dimensional homogeneous spaces  $V_n$ ; equipped with a formal power series*

$$Y(v, z) = \sum_{n \in \mathbf{Z}} v_n z^{-n-1} \in \text{End}(V)[[z, z^{-1}]]$$

*called the vertex operator of  $v$  for each  $v \in V$  satisfying the following (1)  $\sim$  (3).*

*(1) There is a specific element  $\mathbf{1} \in V_0$  called the vacuum such that*

$$(1.1) \ Y(\mathbf{1}, z) = 1_V \text{ and}$$

$$(1.2) \ v_{-1}\mathbf{1} = v \text{ and } v_n\mathbf{1} = 0 \text{ for all } n \geq 0.$$

*(2) There is an specific element  $\mathbf{w} \in V_2$  called the Virasoro element such that*

$$(2.1) \ \{L(n) := \mathbf{w}_{n+1}\} \text{ is a Virasoro algebra generator, that is, they satisfy}$$

$$[L(m), L(n)] = (m - n)L(m + n) + \delta_{m+n,0} \frac{m^3 - m}{12} c, \quad (3.3)$$

*where  $c \in \mathbf{C}$  is called the rank (or the central charge) of  $V$ ,*

*(2.2) the derivation:*

$$[L(-1), Y(v, z)] = \frac{d}{dz} Y(v, z) \quad (3.4)$$

$$(2.3) \ L(0)_{V_n} = n1_{V_n}.$$

*(3) The commutativity:*

$$(z - w)^N Y(v, z) Y(u, w) = (z - w)^N Y(u, w) Y(v, z) \quad (3.5)$$

*for a sufficiently large integer  $N$ .*

**Remark 1** *An important property of vertex operator is the associativity:*

$$Y(v_n u, z) = Y(v, z)_n Y(u, z). \quad (3.6)$$

*Namely, if we are given vertex operators of  $u$  and  $v$ , then vertex operators of  $v_n u$  are determined by them. Another important property is the skew-symmetry:*

$$Y(u, z)v = e^{zL(-1)} Y(v, -z)u. \quad (3.7)$$

*Namely, if given the actions of  $u$  on  $v$ , then we obtain those of  $v$  on  $u$ . One more property we will use is a nondegenerated invariant bilinear form  $(\ , \ )$  satisfying:*

$$(v_m u, w) = (u, v_{2k-m-2} w) \quad (3.8)$$



for  $u, v, w \in V$  satisfying  $L(0)v = kv$  and  $L(r)v = 0$  for all  $r > 0$ . Since we will treat a VOA  $V = \oplus_{i=0} V_n$  with  $\dim V_0 = 1$ , there is a unique invariant bilinear form satisfying  $(\mathbf{w}, \mathbf{w}) = \text{Rank}(V)/2$  for the Virasoro element  $\mathbf{w}$  of  $V$ , see [L1].

**Definition 4** A module for  $(V, Y, \mathbf{1}, \mathbf{w})$  is a  $\mathbf{Z}$ -graded vector space  $M = \oplus_{n \geq 0} M_n$  with finite dimensional homogeneous spaces  $M_n$ ; equipped with a formal power series

$$Y^M(v, z) = \sum_{n \in \mathbf{Z}} v_n^M z^{-n-1} \in (\text{End}(M))[[z, z^{-1}]]$$

called the module vertex operator of  $v$  for  $v \in V$  satisfying:

- (1)  $Y^M(\mathbf{1}, z) = 1_M$ .
- (2)  $Y^M(\mathbf{w}, z) = \sum L^M(n) z^{-n-1}$  satisfies:
  - (2.1) the Virasoro algebra relations,
  - (2.2) the derivation:

$$Y^M(L(-1)v, z) = \frac{d}{dz} Y^M(v, z), \text{ and} \quad (3.9)$$

- (2.3)  $L^M(0)_{M_n} = (k_n)1_{M_n}$  for some  $k_n \in \mathbf{C}$ .

- (3) The commutativity.
- (4) The associativity:

$$Y(u_n v, z) = Y^M(u, z)_n Y^M(v, z). \quad (3.10)$$

## 3.2 Intertwining operator

**Definition 5** Let  $(V, Y, \mathbf{1}, \mathbf{w})$  be a VOA and let  $(W^1, Y^1)$ ,  $(W^2, Y^2)$  and  $(W^3, Y^3)$  be three  $V$ -modules. An intertwining operator of type  $\begin{pmatrix} W^1 \\ W^2 & W^3 \end{pmatrix}$  is a linear map

$$\begin{aligned} I(*, z) : W^2 &\rightarrow (\text{Hom}(W^3, W^1))\{z\} \\ u &\rightarrow I(u, z) = \sum_{n \in \mathbf{Q}} u_n z^{-n-1} \end{aligned}$$

satisfying:

- (1) Derivation:

$$I(L^1(-1)u, z) = \frac{d}{dz} I(u, z). \quad (3.11)$$

- (2) The commutativity: for  $v \in V, u \in W^2$ ,

$$(z - z_1)^N \{Y^1(v, z)I(u, z_1) - I(u, z_1)Y^3(v, z)\} = 0 \quad (3.12)$$

for a sufficiently large integer  $N$ .

(3) The associativity:

$$I(v_n^1 u, z) = Y(v, z)_n I(u, z), \quad (3.13)$$

where the normal product  $Y(v, z)_n I(u, z)$  is given by

$$\text{Res}_{z_1} \{(z_1 - z)^n Y^1(v, z_1) I(u, z) - (-z + z_1)^n I(u, z) Y^3(v, z_1)\} \quad (3.14)$$

and  $Y^i(\mathbf{w}, z) = \sum_{n \in \mathbf{Z}} L^i(n) z^{-n-2}$ .

**Definition 6**  $I_V \left( \begin{smallmatrix} W^1 \\ W^2 & W^3 \end{smallmatrix} \right)$  denotes the set of intertwining operators of type  $\left( \begin{smallmatrix} W^1 \\ W^2 & W^3 \end{smallmatrix} \right)$ . It is a vector space and its dimension is denoted by  $N_{W^2, W^3}^{W^1}$ . In order to denote the dimensions, we use an expression

$$W^2 \times W^3 = \sum_W N_{W^2, W^3}^W W, \quad (3.15)$$

called “fusion rule”, where  $W$  runs over all irreducible  $V$ -modules.

**Definition 7** To simplify the notation, we sometimes omit  $V$  in  $I_V \left( \begin{smallmatrix} W^1 \\ W^2 & W^3 \end{smallmatrix} \right)$ .

We recall the definition of a tensor product from [L2].

**Definition 8** Let  $M^1$  and  $M^2$  be two  $V$ -modules. A pair  $(M, F(*, z))$ , which consists of a  $V$ -module  $M$  and an intertwining operator  $F(*, z)$  of type  $\left( \begin{smallmatrix} M \\ M^1 & M^2 \end{smallmatrix} \right)$ , is called a tensor product for the ordered pair  $(M^1, M^2)$  if the following universal property holds: For any  $V$ -module  $W$  and any intertwining operator  $I(\cdot, z)$  of type  $\left( \begin{smallmatrix} U \\ M^1 & M^2 \end{smallmatrix} \right)$ , there exists a unique  $V$ -homomorphism  $\psi$  from  $M$  to  $U$  such that  $I(*, z) = \psi \cdot F(*, z)$ . Here  $\psi$  extends canonically to a linear map from  $M\{z\}$  to  $U\{z\}$  and  $U\{z\}$  denotes the set of all formal power series  $\sum_{n \in \mathbf{C}} u_n z^n$  with  $u_n \in U$ .

**Remark 2** We should note that we can't define a tensor product of two modules generally. However, since we will treat only rational VOAs in this paper, the tensor products of two modules  $M^1$  and  $M^2$  are always well-defined and it is isomorphic to  $\oplus_U N_{M^1, M^2}^U U$ , where  $U$  runs over the all irreducible  $V$ -modules. Therefore, it is equal to the fusion rule in our case and so we will use the same notation  $M^1 \times M^2$  to denote the tensor product. It is known that  $M^1 \times M^2 \cong M^2 \times M^1$ .

### 3.3 tensor products of VOAs

Let  $(V^1, Y^1, \mathbf{1}^1, \mathbf{w}^1)$  and  $(V^2, Y^2, \mathbf{1}^2, \mathbf{w}^2)$  be VOAs. For  $v^1 \otimes v^2 \in V^1 \otimes V^2$ , define the vertex operator

$$(Y^1 \otimes Y^2)(v^1 \otimes v^2, z) = Y^1(v^1, z) \otimes Y^2(v^2, z) \in \text{End}(V^1 \otimes V^2)[[z, z^{-1}]]$$

of  $v^1 \otimes v^2$  and extend it linearly to the whole space  $V^1 \otimes V^2$ , then

$$(V^1 \otimes V^2, Y^1 \otimes Y^2, \mathbf{1}^1 \otimes \mathbf{1}^2, \mathbf{w}^1 \otimes \mathbf{1}^2 + \mathbf{1}^1 \otimes \mathbf{w}^2) \quad (3.16)$$

becomes a VOA. We will call it the tensor product of  $V^1$  and  $V^2$ .

It is known in [DMZ] and [FHL] that irreducible  $V^1 \otimes V^2$ -modules are tensor products of each irreducible modules.

We will study the fusion rules of  $V^1 \otimes V^2$ -modules. Assume that  $V^1$  and  $V^2$  are rational, that is, they have only finitely many irreducible modules and their modules are direct sums of irreducible submodules.

**Theorem 3.1** *Assume that  $V$  and  $W$  are rational VOAs. Let  $V^1, V^2, V^3$  be irreducible  $V$ -modules and  $W^1, W^2, W^3$  irreducible  $W$ -modules such that  $N_{V^1}^{V^3}{}_{V^2} \leq 1$ . Then*

$$I_V \begin{pmatrix} V^3 \\ V^1 & V^2 \end{pmatrix} \otimes I_W \begin{pmatrix} W^3 \\ W^1 & W^2 \end{pmatrix} = I_{V \otimes W} \begin{pmatrix} V^3 \otimes W^3 \\ V^1 \otimes W^1 & V^2 \otimes W^2 \end{pmatrix} \quad (3.17)$$

**[Proof]** We will use the same argument as in the proof of Proposition 4.4 in [M1]. Clearly, we have

$$I \begin{pmatrix} V^3 \\ V^1 & V^2 \end{pmatrix} \otimes I \begin{pmatrix} W^3 \\ W^1 & W^2 \end{pmatrix} \subseteq I \begin{pmatrix} V^3 \otimes W^3 \\ V^1 \otimes W^1 & V^2 \otimes W^2 \end{pmatrix}.$$

We will show the reverse. Take

$$I(*, z) \in I \begin{pmatrix} V^3 \otimes W^3 \\ V^1 \otimes W^1 & V^2 \otimes W^2 \end{pmatrix}.$$

Set  $T = V \otimes W$  and view  $V$  and  $W$  as subVOAs  $V \otimes \mathbf{1}_W$  and  $\mathbf{1}_V \otimes W$ , respectively. Let  $e$  and  $f$  be Virasoro elements of  $V$  and  $W$ . Since  $V$  is rational,  $V^i \otimes W^i$  are direct sums of irreducible  $V$ -modules. Let  $M^1, M^2, M^3$  be irreducible  $V$ -submodules of  $V^1 \otimes W^1, V^2 \otimes W^2$ , and  $V^3 \otimes W^3$ , respectively. Clearly,  $M^i$  is isomorphic to  $V^i$  as  $V$ -modules for  $i = 1, 2, 3$ . Let  $\pi$  denote the projection map  $\pi : V^3 \otimes W^3 \rightarrow M^3$  such that  $\pi|_{M^3} = 1_{M^3}$ . Since  $f_1$  acts on  $W^i$  diagonally and the actions of  $f_1$  on  $V^i \otimes W^i$  commutes with the

actions of  $V$  on  $V^i \otimes W^i$ , we may assume that  $f_1$  acts on  $M^i$  as scalars  $\alpha_i \in \mathbf{C}$ . Let  $u \in M^1$ ,  $v \in M^2$ . Since

$$\begin{aligned} f_1^3 u_{m-1} v &= \{f_1^3 u_{m-1} - u_{m-1} f_1^2\} v + u_{m-1} f_1^2 v \\ &= (f_0 u)_m v + (f_1 u)_{m-1} v + u_{m-1} \alpha^2 v \\ &= (f_0 u)_m v + (\alpha^1 + \alpha^2) u_{m-1} v, \end{aligned}$$

we get

$$(f_0 u)_m v = (f_1 - \alpha^1 - \alpha^2) u_{m-1} v$$

and also

$$(f_0 u)_m v = ((L(-1) - e_0)u)_m v = (L(-1)v)_m u - (e_0 v)_m v = -m u_{m-1} v - (e_0 u)_m v.$$

We hence have

$$\pi((f_0 u)_m v) = (\alpha^3 - \alpha^1 - \alpha^2) \pi(u_{m-1} u).$$

For simplicity, set  $\alpha = \alpha(M_2, M_3) = \alpha^3 - \alpha^1 - \alpha^2$ . Then we obtain

$$\pi((e_0 u)_m v) = \pi(((L(-1) - f_0)u)_m v) = (-m - \alpha) \pi(u_{m-1} v).$$

If we set  $I^1(u, z) = \pi(I(u, z))z^{-\alpha}$ , then it is easy to see that  $I^1(*, z)$  satisfies the Jacobi identity. Also, the above equation implies the  $e_0$ -derivative formula:

$$I(e_0 u, z) v = \left(\frac{d}{dz} I^1(u, z)\right) v.$$

Hence,  $I^1(*, z)$  is an intertwining operator of  $V$  of type  $\begin{pmatrix} V^3 \\ V^1 & V^2 \end{pmatrix}$ . We fix an intertwining operator  $I^1(*, z)$  of type  $\begin{pmatrix} V^3 \\ V^1 & V^2 \end{pmatrix}$ . By the assumption  $N_{r^1}^{r^3} r^2 \leq 1$ , we have an expression  $I(u, z) = B(w) \otimes I^1(u, z)$  for  $w \in W^1$  and  $u \in M^1 = V \otimes w$ , where  $B(w) = (\lambda_{M^2, M^3} z^{\alpha(M^2, M^3)})$  is an infinite dimensional matrix, where  $M^2$  and  $M^3$  run over all irreducible components of the direct sums of  $V^2 \otimes W^2$  and  $V^3 \otimes W^3$ . In particular, we may view  $B(w)$  as an element of  $\text{Hom}(W^2, W^3)\{z, z^{-1}\}$ . Since  $I(*, z)$ ,  $I^1(*, z)$  satisfy the commutativity, the associativity, and the derivation, so does  $B(w)$ . Hence we conclude that  $B(w) = I^2(w, z)$  is an intertwining operator of  $W$  and  $I(u \otimes w, z) = I^1(u, z) \otimes I^2(w, z)$  as desired.  $\blacksquare$

As a corollary, we will show the following:

**Corollary 3.1** *Assume that  $V^1$  and  $V^2$  are rational VOAs, then  $V^1 \otimes V^2$  is rational.*

**[Proof]** Set  $V = V^1 \otimes V^2$ . Suppose false and let  $(W, Y^W)$  be an indecomposable  $V$ -module. Since  $V$  has only finite number of nonisomorphic irreducible modules by [DMZ],  $W$  has an irreducible  $V$ -submodule  $W^0 = U^1 \otimes U^2$ . Since  $V^i \times U^i \cong U^i$  for  $i = 1, 2$ , we have that all irreducible factors of  $W$  are isomorphic to  $W^0$  and hence  $W$  has a series  $0 \subseteq W^0 \subseteq \dots \subseteq W^k = W$  such that  $W^i/W^{i-1} \cong W^0$  as  $V$ -modules. In particular,  $W^i = W^{i-1} \oplus X^i$  splits as a vector space and  $W = \oplus_{i=0}^k X^i$  as vector spaces. Let  $\pi$  be a projection  $\pi^i : W^i \rightarrow X^i$  such that  $\pi(W^{i-1}) = 0$ . Then for  $v^1 \in V^1$ ,  $v^2 \in V^2$ , by taking the restriction into  $X^i$  and the image of  $\pi^j$  of the module vertex operator, we have

$$\pi^j Y(v^1 \otimes v^2, z) : X^i \rightarrow X^j \in \text{Home}(X^i, X^j) U^j[[z, z^{-1}]].$$

It follows from the properties of the module vertex operator that the above is an intertwining operator of type  $\begin{pmatrix} U^1 \otimes U^2 \\ V^1 \otimes V^2 & U^1 \otimes U^2 \end{pmatrix}$ . Since  $V^1 \times W^0 = W^0$  and  $V^2 \times U^0 = U^0$ , we have  $\dim I \begin{pmatrix} U^1 \otimes U^2 \\ V^1 \otimes V^2 & U^1 \otimes U^2 \end{pmatrix} = 1$  by the previous theorem. Let  $I(*, z) \in I \begin{pmatrix} U^1 \otimes U^2 \\ V^1 \otimes V^2 & U^1 \otimes U^2 \end{pmatrix}$  be a nonzero intertwining operator such that  $I(\mathbf{1}, z) = id_{U^1 \otimes U^2}$ . Since  $W = \oplus_{i=0}^k X^i$ , we have the following expression:

$$Y(v, z) = A \otimes I(v, z),$$

for some  $(k+1) \times (k+1)$ -matrix  $A$ . Since  $Y(\mathbf{1}, z) = 1_W$ ,  $A$  is the identity matrix and  $W$  is a direct sum of irreducible  $V^1 \otimes V^2$ -modules. ■

## 4 Tensor product of SVOAs

In this section, we recall the results from [M2] and construct a VOA  $M_D$  from an even binary linear code  $D$ .

### 4.1 Minimal SVOA M

Let us construct first a vertex operator superalgebra (SVOA)

$$M = L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2}). \quad (4.1)$$

Let  $L = \mathbf{Z}x$  be a lattice with  $\langle x, x \rangle = 1$ . Viewing  $H = \mathbf{C}x$  as a commutative Lie algebra with a bilinear form  $\langle, \rangle$ , we define the affine Lie algebra

$$\hat{H} = H[t, t^{-1}] + \mathbf{C}k$$

associated with  $H$  and the symmetric tensor algebra  $M(1)$  of  $\hat{H}$ . As they did in [FLM], we shall define the Fock space  $V_L = \oplus_{a \in L} M(1)e^a$  with the vacuum  $\mathbf{1} = e^0$  and the vertex operators  $Y(*, z)$  as follows:

The vertex operator of  $e^a$  is given by

$$Y(e^a, z) = \exp \left( \sum_{n \in \mathbf{Z}_+} \frac{a(-n)}{n} z^n \right) \exp \left( \sum_{n \in \mathbf{Z}_+} \frac{a(n)}{-n} z^{-n} \right) e^a z^a. \quad (4.2)$$

and that of  $a(-1)e^0$  is

$$Y(a(-1)e^0, z) = \sum a(n) z^{-n-1}.$$

The vertex operators of other elements are defined by the normal product:

$$Y(a(n)v, z) = Y(a(-1)e^0, z)_n Y(v, z). \quad (4.3)$$

By the direct calculation as they did in §4 in [FLM], we have:

**Lemma 4.1**

$$\begin{aligned} [\sum a(n)x^{-n-1}, Y(e^b, z)] &= 0 && \text{for any } a, b \in L, \\ (x-z)^N [Y(e^a, x), Y(e^b, z)] &= 0 && \text{for } \langle a, b \rangle \equiv 2 \pmod{2}, \\ (x-z)^N \{Y(e^a, x)Y(e^b, z) + Y(e^b, z)Y(e^a, z)\} &= 0 && \text{for } \langle a, b \rangle \equiv 1 \pmod{2}, \end{aligned} \quad (4.4)$$

for a sufficiently large integer  $N$ . These are called "supercommutativity".

**Remark 3** We note that the above relations hold generally. Namely, if we define a module vertex operator  $I^W$  on  $W = M(1) \otimes e^{\frac{1}{2}\mathbf{Z}x}$  by

$$I^W(e^a, z) = \exp \left( \sum_{n \in \mathbf{Z}_+} \frac{a(-n)}{n} z^n \right) \exp \left( \sum_{n \in \mathbf{Z}_+} \frac{a(n)}{-n} z^{-n} \right) e^a z^a \quad (4.5)$$

and

$$I^W(a(-1)\mathbf{1}, z) = \sum a(n) z^{-n-1}, \quad (4.6)$$

then they satisfy the same super-commutativity. We should note that if  $a \notin 2\mathbf{Z}x$ , then the powers of  $z$  in  $I^W(e^a, z)$  are not integers since  $z^x e^{x/2} = z^{\frac{1}{2}}$ . Namely,  $I^M(e^a, z)$  does not satisfy one of the conditions of module vertex operators.

In [M2], we found two mutually orthogonal conformal vectors

$$\begin{aligned} \mathbf{w}^1 &= \frac{1}{4}x(-1)^2\mathbf{1} + \frac{1}{4}(e^{2x} + e^{-2x}) && \text{and} \\ \mathbf{w}^2 &= \frac{1}{4}x(-1)^2\mathbf{1} - \frac{1}{4}(e^{2x} + e^{-2x}) \end{aligned} \quad (4.7)$$

with central charge  $\frac{1}{2}$  such that  $\mathbf{w} = \mathbf{w}^1 + \mathbf{w}^2 = \frac{1}{2}x(-1)^2\mathbf{1}$  is the Virasoro element of  $V_L$ , where  $\mathbf{1} = e^0$  is the vacuum. We also note that  $\mathbf{w}^1$  and  $\mathbf{w}^2$  are rational since  $V_L$  has a positive definite invariant bilinear form induced from the inner product of  $L$ . As we mentioned at the beginning,  $\langle \mathbf{w}^i \rangle \cong L(\frac{1}{2}, 0)$  has only three irreducible modules  $L(\frac{1}{2}, 0), L(\frac{1}{2}, \frac{1}{2}), L(\frac{1}{2}, \frac{1}{16})$ . By calculating the dimensions of weight spaces, we conclude that  $V_L$  is isomorphic to the tensor product  $(L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2})) \otimes (L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2}))$  as  $\langle \mathbf{w}^1 \rangle \otimes \langle \mathbf{w}^2 \rangle$ -modules, where  $\langle \mathbf{w}^i \rangle$  denotes the vertex operator subalgebra generated by  $\mathbf{w}^i$ . Let  $\theta$  be the automorphism of  $V_L$  induced from the automorphism  $-1$  on  $L$ . Take the fixed point space  $(V_L)^\theta$  of  $V_L$  by  $\theta$ . This is also a SVOA containing  $\mathbf{w}^1$  and  $\mathbf{w}^2$  and we obtain the decomposition:

$$(V_L)^\theta \cong \left( L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0) \right) \oplus \left( L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{1}{2}, 0) \right) \quad (4.8)$$

as  $\langle \mathbf{w}^1 \rangle \otimes \langle \mathbf{w}^2 \rangle$ -modules by calculating the dimensions of weight spaces. Take the subspace  $M = \{v \in (V_L)^\theta \mid \mathbf{w}^2_1 v = 0\}$ . Then  $M$  is a SVOA with the Virasoro element  $\mathbf{w}^1$  and we see  $M \cong L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2})$  as  $\langle \mathbf{w}^1 \rangle$ -modules.

Therefore, we proved the following theorem (see [M2]).

**Theorem 4.1**  *$(M, Y, \mathbf{w}^1, e^0)$  is a simple SVOA with the even part  $M_{\overline{0}} \cong L(\frac{1}{2}, 0)$  and the odd part  $M_{\overline{1}} \cong L(\frac{1}{2}, \frac{1}{2})$  as  $L(\frac{1}{2}, 0)$ -modules. The central charge of  $(M, Y)$  is  $\frac{1}{2}$  and it has a positive definite invariant bilinear form. Here we define the vertex operator  $Y$  by restricting  $V_L$  into  $M$ .*

For the later use, we fix a lowest weight vector

$$q = \frac{1}{\sqrt{2}}(e^x + e^{-x}) \in M_{\overline{1}} \quad (4.9)$$

of the odd part. By the direct calculation, we obtain

$$q_{-2}q = 2\mathbf{w}^1, \quad q_{-1}q = 0, \quad q_0q = \mathbf{1}. \quad (4.10)$$

It is easy to check the following correspondences in  $V_{\mathbf{Z}a}$ :

$$\begin{aligned} x(-1)e^0 &\in L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{1}{2}, \frac{1}{2}), \\ \frac{1}{\sqrt{2}}(e^x + e^{-x}) &\in L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{1}{2}, 0), \text{ and} \\ \frac{1}{\sqrt{2}}(e^x - e^{-x}) &\in L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, \frac{1}{2}). \end{aligned} \quad (4.11)$$

We also get  $\frac{1}{2}(e^x + e^{-1})_{-1}(e^x - e^{-1}) = x(-1)e^0$ .

We can construct a  $V_{2\mathbf{Z}x}$ -module  $W = V_{\mathbf{Z}x/2}$  and the module vertex operator  $Y^W$  by the same ways. As we mentioned in Remark 3, we have a formal power series  $I^W(v, z) \in \text{End}(W)[[z^{1/2}, z^{-1/2}]]$  for  $v \in V_{\mathbf{Z}x}$ . Clearly, for  $v \in V_{2\mathbf{Z}x}$ ,  $I^W(v, z) = Y^W(v, z)$  and hence  $I^W(q, z) = \sum q_n z^{-n-1}$  satisfies the commutativity with  $Y^W(u, z)$  of  $u \in V_{2\mathbf{Z}x}$ . Namely,  $I^W(*, z)$  is an intertwining operator of type  $\begin{pmatrix} W \\ M & W \end{pmatrix}$ . The  $V_{2\mathbf{Z}x}$ -module  $V_{\mathbf{Z}x/2}$  splits up into a direct sum

$$V_{\mathbf{Z}x/2} = V_{\mathbf{Z}x} \oplus V_{2\mathbf{Z}x+x/2} \oplus V_{2\mathbf{Z}x-x/2}$$

of irreducible  $V_{2\mathbf{Z}x}$ -modules and  $q_n$  permutes  $\{V_{2\mathbf{Z}x+x/2}, V_{2\mathbf{Z}x-x/2}\}$  for any  $n = \frac{1}{2} + \mathbf{Z}$ . By the direct calculations, we see that  $V_{2\mathbf{Z}x+x/2}$  and  $V_{2\mathbf{Z}x-x/2}$  are both isomorphic to  $L(\frac{1}{2}, \frac{1}{16}) \otimes L(\frac{1}{2}, \frac{1}{16})$  as  $\langle \mathbf{w}^1 \rangle \otimes \langle \mathbf{w}^2 \rangle$ -modules. Fix the lowest weight vectors  $e^{\frac{1}{2}x}$  and  $e^{-\frac{1}{2}x}$  of  $V_{2\mathbf{Z}x+x/2}$  and  $V_{2\mathbf{Z}x-x/2}$ , respectively. By restricting  $v$  in  $M_1 \cong L(\frac{1}{2}, \frac{1}{2})$  and taking the eigenspaces of  $\mathbf{w}^2$  with an eigenvalue  $\frac{1}{16}$ ,  $I^W(v, z)$  defines the following three intertwining operators:

$$\begin{aligned} I^{\frac{1}{2}, 0}(*, z) &\in I \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \\ I^{\frac{1}{2}, \frac{1}{2}}(*, z) &\in I \begin{pmatrix} 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \text{ and} \\ I^{\frac{1}{2}, \frac{1}{16}}(*, z) &\in I \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} & \frac{1}{16} \end{pmatrix}. \end{aligned} \tag{4.12}$$

Also, the restriction to  $M_0 \cong L(\frac{1}{2}, 0)$  defines the following intertwining operators:

$$\begin{aligned} I^{0, 0}(*, z) &\in I \begin{pmatrix} 0 \\ 0 & 0 \end{pmatrix}, \\ I^{0, \frac{1}{2}}(*, z) &\in I \begin{pmatrix} \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} \text{ and} \\ I^{0, \frac{1}{16}}(*, z) &\in I \begin{pmatrix} \frac{1}{2} \\ 0 & \frac{1}{16} \end{pmatrix}, \end{aligned} \tag{4.13}$$

which are actually module vertex operators of  $\langle \mathbf{w}^1 \rangle$ . We fix these intertwining operators throughout this paper.

Using the above intertwining operators, the formal power series



$Y(u \otimes v, z) \in \text{End}(V_{(\frac{1}{2}+\mathbf{Z})x})\{z, z^{-1}\}$  given by (4.2) and (4.5) are

$$\begin{aligned}
Y(u \otimes v, z) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes I^{0, \frac{1}{16}}(u, z) \otimes I^{0, \frac{1}{16}}(v, z) & \text{for } u \otimes v \in L(\tfrac{1}{2}, 0) \otimes L(\tfrac{1}{2}, 0), \\
Y(u \otimes v, z) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes I^{\frac{1}{2}, \frac{1}{16}}(u, z) \otimes I^{\frac{1}{2}, \frac{1}{16}}(v, z) & \text{for } u \otimes v \in L(\tfrac{1}{2}, \tfrac{1}{2}) \otimes L(\tfrac{1}{2}, \tfrac{1}{2}), \\
Y(u \otimes v, z) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes I^{\frac{1}{2}, \frac{1}{16}}(u, z) \otimes I^{0, \frac{1}{16}}(v, z) & \text{for } u \otimes v \in L(\tfrac{1}{2}, \tfrac{1}{2}) \otimes L(\tfrac{1}{2}, 0), \\
Y(u \otimes v, z) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes I^{0, \frac{1}{16}}(u, z) \otimes I^{\frac{1}{2}, \frac{1}{16}}(v, z) & \text{for } u \otimes v \in L(\tfrac{1}{2}, 0) \otimes L(\tfrac{1}{2}, \tfrac{1}{2}).
\end{aligned} \tag{4.14}$$

**Remark 4** By the definitions (4.2) and (4.5) of  $Y(*, z)$  and  $I^W(*, z)$ , we have:

- (1) The powers of  $z$  in  $I^{0,*}(*, z)$ ,  $I^{\frac{1}{2}, 0}(*, z)$  and  $I^{\frac{1}{2}, \frac{1}{2}}(*, z)$  are all integers and those of  $z$  in  $I^{\frac{1}{2}, \frac{1}{16}}(*, z)$  are half-integers, that is, in  $\frac{1}{2} + \mathbf{Z}$ .
- (2)  $I^W(*, z)$  satisfies the derivation.
- (3)  $I^W(*, z)$  satisfies the supercommutativity:

$$\begin{aligned}
I^W(v, z_1)I^W(v', z_2) &\sim I^W(v', z_2)I^W(v, z_1), \\
I^W(v, z_1)I^W(u, z_2) &\sim I^W(u, z_2)I^W(v, z_1), \\
I^W(u, z_1)I^W(u', z_2) &\sim (-1)I^W(u', z_2)I^W(u, z_1),
\end{aligned} \tag{4.15}$$

for  $v, v' \in M_0$  and  $u, u' \in M_1$ . Here  $A(z_1, z_2) \sim B(z_1, z_2)$  implies that

$$(z_1 - z_2)^N A(z_1, z_2) = (z_1 - z_2)^N B(z_1, z_2)$$

for sufficiently large integer  $N$ , see Proposition 4.3.2 in [FLM]. In particular,  $I^{*, \frac{1}{16}}(*, z)$  satisfies the supercommutativity:

$$\begin{aligned}
I^{0, \frac{1}{16}}(v, z_1)I^{0, \frac{1}{16}}(v', z_2) &\sim I^{0, \frac{1}{16}}(v', z_2)I^{0, \frac{1}{16}}(v, z_1), \\
I^{0, \frac{1}{16}}(v, z_1)I^{\frac{1}{2}, \frac{1}{16}}(u, z_2) &\sim I^{\frac{1}{2}, \frac{1}{16}}(u, z_2)I^{0, \frac{1}{16}}(v, z_1), \\
I^{\frac{1}{2}, \frac{1}{16}}(u, z_1)I^{\frac{1}{2}, \frac{1}{16}}(u', z_2) &\sim -I^{\frac{1}{2}, \frac{1}{16}}(u', z_2)I^{\frac{1}{2}, \frac{1}{16}}(u, z_1),
\end{aligned} \tag{4.16}$$

for  $v, v' \in M_0$  and  $u, u' \in M_1$ .

We next show an associativity.

**Remark 5** Clearly,  $V_{2\mathbf{Z}x} = \{L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0)\} \oplus \{L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{1}{2}, \frac{1}{2})\}$  and it becomes a vertex operator algebra and  $W = V_{\frac{1}{2}x+2\mathbf{Z}x}$  is a  $V_{2\mathbf{Z}x}$ -module as we showed. Viewing

$L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0)$ -module, the module vertex operator  $Y^W$  has the following structure by (4.14).

$$(I \otimes I)(u \otimes v, z) = \begin{cases} I^{0, \frac{1}{16}}(u, z) \otimes I^{0, \frac{1}{16}}(v, z) & \text{for } u, v \in L(\frac{1}{2}, 0), \\ I^{\frac{1}{2}, \frac{1}{16}}(u, z) \otimes I^{\frac{1}{2}, \frac{1}{16}}(v, z) & \text{for } u, v \in L(\frac{1}{2}, \frac{1}{2}). \end{cases} \quad (4.17)$$

In particular,  $I \otimes I$  satisfy the associativity and the derivation.

For example, for  $u, v, u', v' \in L(\frac{1}{2}, \frac{1}{2})$ ,

$$\begin{aligned} & (I^{0, \frac{1}{16}} \otimes I^{0, \frac{1}{16}})((u \otimes v)_n(u' \otimes v'), z) \\ &= ((I^{\frac{1}{2}, \frac{1}{16}} \otimes I^{\frac{1}{2}, \frac{1}{16}})(u \otimes v, z))_n((I^{\frac{1}{2}, \frac{1}{16}} \otimes I^{\frac{1}{2}, \frac{1}{16}})(u' \otimes v', z)). \end{aligned}$$

It is known that  $Y(*, z)$  satisfies :

$$Y(x(n)v, z) = Y(x(-1)\mathbf{1}, z)_n Y(v, z).$$

Namely,  $Y(*, z)$  satisfies the associativity with the actions of  $x(-1)\mathbf{1} \in L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{1}{2}, \frac{1}{2})$ .

By the direct calculation, we see that the normal product satisfies the associativity:

$$(a(z)_n b(z))_m c(z) = \sum_{i=0}^{\infty} (-1)^i \binom{n}{i} (a(z)_{n-i} b(z)_{m+i} c(z) - (-1)^n b(z)_{n+m-i} a(z)_i c(z)).$$

Since  $Y(*, z)$  satisfies the associativity with the actions of  $L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0)$ ,  $Y(*, z)$  satisfies the associativity with the actions of  $L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{1}{2}, \frac{1}{2})$ . Therefore, for  $u^1 \otimes u^2 \in L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{1}{2}, \frac{1}{2})$  and  $u^3 \otimes v \in L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{1}{2}, 0)$ , we calculate on the actions on  $V_{(\frac{1}{2}+\mathbf{Z})_x}$  as follows:

$$\begin{aligned} & Y((u^1 \otimes u^2)_n(u^3 \otimes v), z) \\ &= Y(u^1 \otimes u^2, z)_n Y(u^3 \otimes v, z) \\ &= \text{Res}_x \{ (x-z)^n Y(u^1 \otimes u^2, x) Y(u^3 \otimes v, z) - (-z+x)^n Y(u^3 \otimes v, z) Y(u^1 \otimes u^2, x) \} \\ &= \text{Res}_x \{ (x-z)^n \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes (I \otimes I)(u^1 \otimes u^2, x) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes (I \otimes I)(u^3 \otimes v, z) \\ & \quad - (-z+x)^n \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes (I \otimes I)(u^3 \otimes v, z) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes (I \otimes I)(u^1 \otimes u^2, x) \} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{Res}_x \{ (x-z)^n \otimes I \otimes I(u^1 \otimes u^2, x) \otimes I \otimes I(u^3 \otimes v, z) \\ & \quad + (-z+x)^n \otimes I \otimes I(u^3 \otimes v, z) \otimes I \otimes I(u^1 \otimes u^2, x) \}. \end{aligned}$$

On the other hand, since  $(u^1 \otimes u^2)_n(u^3 \otimes v) \in L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, \frac{1}{2})$ , we get

$$\begin{aligned} & Y((u^1 \otimes u^2)_n(u^3 \otimes v), z) \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes (I \otimes I)((u^1 \otimes u^2)_n(u^3 \otimes v), z). \end{aligned}$$

Since  $I^{*, \frac{1}{16}}(*, z)$  satisfies the associativity with the actions of  $L(\frac{1}{2}, 0)$ , we obtain :

$$I^{0, \frac{1}{16}}(u_n v, z) = I^{\frac{1}{2}, \frac{1}{16}}(u, z)_n I^{\frac{1}{2}, \frac{1}{16}}(v, z)$$

for  $u, v \in L(\frac{1}{2}, \frac{1}{2})$ , where the normal product of  $I^{\frac{1}{2}, \frac{1}{16}}(*, z)$  and  $I^{\frac{1}{2}, \frac{1}{16}}(*, z)$  is given by

$$a(z)_n b(z) = \text{Res}_x \{ (x - z)^n a(x) b(z) + (-z + x)^n b(z) a(x) \}$$

and we will call this normal product "super normal product" and the above associativity "superassociativity". Namely,  $I^{h, \frac{1}{16}}(*, z)$  and  $I^{k, \frac{1}{16}}(*, z)$  satisfies the superassociativity:

$$\begin{aligned} & I^{h+k, \frac{1}{16}}(u_n v, z) \\ &= \text{Res}_x \{ (x - z)^n I^{h, \frac{1}{16}}(u, x) I^{k, \frac{1}{16}}(v, z) - (-1)^{|u||v|} (-z + x)^n I^{k, \frac{1}{16}}(v, z) I^{h, \frac{1}{16}}(u, x) \} \end{aligned} \quad (4.18)$$

for  $u \in L(\frac{1}{2}, h), v \in L(\frac{1}{2}, k)$  and  $h, k = 0, \frac{1}{2}$ , where  $|v| = 0$  if  $v \in L(\frac{1}{2}, 0)$  and  $|v| = 1$  if  $v \in L(\frac{1}{2}, \frac{1}{2})$ .

Let extend it into the tensor product of many  $M_0$  and  $M_1$ . Using the tensor product and (4.18), we obtain the following:

**Proposition 4.1** For two binary words  $(a^i), (b^i)$  of even length  $n$  and  $u \in \otimes_{i=1}^n L(\frac{1}{2}, \frac{a^i}{2})$ ,  $v \in \otimes_{i=1}^n L(\frac{1}{2}, \frac{b^i}{2})$ ,  $\otimes I$  satisfies the superassociativity and the derivation:

$$\begin{aligned} & (\otimes I^{(a^i+b^i)/2, \frac{1}{16}})(u_n v, z) = \left( (\otimes I^{a^i/2, \frac{1}{16}})(u, z) \right)_n \left( (\otimes I^{b^i/2, \frac{1}{16}})(v, z) \right) \\ & [L(-1), (\otimes I^{a^i/2, \frac{1}{16}})(u, z)] = \frac{d}{dz} \left\{ (\otimes I^{a^i/2, \frac{1}{16}})(u, z) \right\}. \end{aligned} \quad (4.19)$$

We often abuse the notation  $0, 1 \in \mathbf{Z}_2$  to denote integers 0, 1. For example,  $a_i/2 = 0$  for  $a_i = 0$  and  $a_i/2 = \frac{1}{2}$  for  $a_i = 1$ .

**[Proof]** By reordering the coordinates, we may assume  $(a^i) = (1^{2s} 0^{n-2s})$ . Dividing the coordinates into a set of pairs  $\{1, 2\} \{3, 4\} \cdots \{2s-1, 2s\} \{2s+1, \dots, n\}$  and then applying the above results to the pair  $\{i, i+1\}$ , we have the superassociativity for each  $(a^i, a^{i+1}) = (1, 1)$  and  $(b^i, b^{i+1})$ . For  $a_i = 0$ , we have the associativity. Taking the tensor product of them, we obtain the desired superassociativity. The derivation comes from Remark 4. ■

## 4.2 Tensor product of SVOA

Let us start a construction of VOA from an even binary linear code  $D$  using the tensor product of vertex operator superalgebras.

In the previous subsection, we constructed the vertex operator superalgebra (SVOA)

$$M = M_{\overline{0}} \oplus M_{\overline{1}} = L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2}), \quad (4.20)$$

where  $M_{\overline{0}} = L(\frac{1}{2}, 0)$  is the even part and  $M_{\overline{1}} = L(\frac{1}{2}, \frac{1}{2})$  is the odd part. For  $v \in M_{\overline{i}}$  and  $i = 0, 1$ ,  $|v|$  denotes  $i$ . We note that the notion of vertex operator superalgebra is given by supercommutativity:

$$Y(v, z_1)Y(w, z_2) \sim (-1)^{|v||w|}Y(w, z_2)Y(v, z_1). \quad (4.21)$$

The tensor product of SVOAs  $(M^1, Y^1), \dots, (M^n, Y^n)$  is given by the following:  
The full Fock space is

$$\hat{M}_F = M^1 \otimes \dots \otimes M^n$$

and we define the vertex operator  $\hat{Y}$  of  $v^1 \otimes \dots \otimes v^n \in M^1 \otimes \dots \otimes M^n$  by

$$\hat{Y}(v^1 \otimes \dots \otimes v^n, z) = Y^1(v^1, z) \otimes \dots \otimes Y^n(v^n, z) \quad (4.22)$$

and extend it to  $\hat{M}_F$  linearly.

In this paper, we will take copies of  $M$  as  $M^i$ , that is,  $M^1 \cong \dots \cong M^n \cong M = L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2})$ .

For a word  $\delta = (d_1, \dots, d_n) \in \mathbf{Z}_2^n$ , set

$$\hat{M}_\delta = \otimes_{i=1}^n (M^i)_{\overline{d_i}}, \quad (4.23)$$

where  $(M^i)_{\overline{0}}$  and  $(M^i)_{\overline{1}}$  are the even part and the odd part of  $M^i \cong L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2})$ , respectively. For example,  $\hat{M}_{(0\dots 0)} \cong L(\frac{1}{2}, 0) \otimes \dots \otimes L(\frac{1}{2}, 0)$  and  $\hat{M}_{(1\dots 1)} \cong L(\frac{1}{2}, \frac{1}{2}) \otimes \dots \otimes L(\frac{1}{2}, \frac{1}{2})$ .

By the definition of  $\hat{M}_\delta$  and the  $\mathbf{Z}_2$ -gradations of SVOAs, we have the following:

**Lemma 4.2** *Let  $\delta = (d_1, \dots, d_n)$  and  $\gamma = (g_1, \dots, g_n)$  be words. For  $v \in \hat{M}_\delta$  and  $w \in \hat{M}_\gamma$ , we obtain*

$$v_m w \in \hat{M}_{\delta+\gamma} \quad \text{for any } m \in \mathbf{Z}. \quad (4.24)$$

Using the supercommutativity (4.15) in SVOA, we have the following lemma.

**Lemma 4.3** *Let  $\delta = (d_1, \dots, d_n)$  and  $\gamma = (g_1, \dots, g_n)$  be words. For  $v \in \hat{M}_\delta$  and  $w \in \hat{M}_\gamma$ , we obtain*

$$\hat{Y}(v, z_1)\hat{Y}(w, z_2) \sim (-1)^{\langle \delta, \gamma \rangle} \hat{Y}(w, z_2)\hat{Y}(v, z_1). \quad (4.25)$$

Here  $\langle \delta, \gamma \rangle = d_1 g_1 + \dots + d_n g_n$ .

*In particular, if  $\delta$  has an even weight, then  $\hat{Y}(v, z)$  satisfies the commutativity with  $\hat{Y}(v, z)$  itself for  $v \in \hat{M}_\delta$ .*

For the purpose of constructing a VOA, we need the commutativity. We shall use a central extension  $\hat{D}$  of  $D$  by  $\pm 1$  in order to modify the supercommutativity of the above vertex operators. Let  $\nu_i$  denote the word  $(0, \dots, 0, 1, 0, \dots, 0)$  whose  $i$ -th entry is one and the other entries are all 0 and let  $e^{\nu_i}$  be formal elements satisfying  $e^{\nu_i} e^{\nu_i} = 1$  and  $e^{\nu_i} e^{\nu_j} = -e^{\nu_j} e^{\nu_i}$  for  $i \neq j$ . For any even word  $\alpha = \nu_{j_1} + \dots + \nu_{j_k}$  with  $j_1 < \dots < j_k$ , set

$$e^\alpha = e^{\nu_{j_1}} e^{\nu_{j_2}} \dots e^{\nu_{j_k}}. \quad (4.26)$$

It is easy to see the following result.

**Lemma 4.4** *For  $\alpha, \beta$ ,*

$$\begin{cases} e^\alpha e^\beta = (-1)^{\langle \alpha, \beta \rangle} e^\beta e^\alpha & \text{if } |\alpha||\beta| \text{ is even} \\ e^\alpha e^\beta = -(-1)^{\langle \alpha, \beta \rangle} e^\beta e^\alpha & \text{if } |\alpha||\beta| \text{ is odd} \end{cases} \quad (4.27)$$

**[Proof]** Since

$$e^\alpha e^{\nu_i} = \begin{cases} (-1)^{|\alpha|} e^{\nu_i} e^\alpha & \text{if } \langle \alpha, \nu_i \rangle = 0 \\ (-1)^{|\alpha|-1} e^{\nu_i} e^\alpha & \text{if } \langle \alpha, \nu_i \rangle = 1, \end{cases} \quad (4.28)$$

we have the desired results. ■

In order to combine (4.27) into the vertex operator  $\hat{Y}$ , set

$$M_D = \oplus_{\delta \in D} (\hat{M}_\delta \otimes e^\delta) \quad (4.29)$$

and define a new vertex operator  $Y$  by

$$Y(v \otimes e^\beta, z) = \hat{Y}(v, z) \otimes e^\beta \quad (4.30)$$

for  $v \otimes e^\beta \in \hat{M}_\beta \otimes e^\beta$ . We then obtain the desired commutativity:

$$Y(v, z_1)Y(w, z_2) \sim Y(w, z_2)Y(v, z_1). \quad (4.31)$$

Let  $\mathbf{w}^i$  be the Virasoro element of  $M^i$ . Then

$$\mathbf{w} = \sum_{i=1}^n (\mathbf{1}^1 \otimes \dots \otimes \mathbf{1}^{i-1} \otimes \mathbf{w}^i \otimes \mathbf{1}^{i+1} \otimes \dots \otimes \mathbf{1}^n) \otimes e^0 \quad (4.32)$$

satisfies the desired properties of the Virasoro element of  $M_D$  and

$$\mathbf{1} = \mathbf{1}^1 \otimes \dots \otimes \mathbf{1}^n \otimes e^0$$

is the vacuum of  $M_D$ , where  $\mathbf{1}^i$  is the vacuum of  $M^i$ . Finally, we have the following theorem.

**Theorem 4.2** *If  $D$  is an even binary linear code, then  $(M_D, Y, \mathbf{w}, \mathbf{1})$  is a VOA.*

By the choice of our cocycle, we easily obtain the following result:

**Theorem 4.3**  *$\text{Aut}(D)$  is an automorphism group of VOA  $(M_D, Y)$ .*

[**Proof**]  $\text{Aut}(D)$  acts on  $\{\hat{M}_\alpha : \alpha \in D\}$  as permutations and also commutes with the products (4.28) in the central extension  $\{\pm e^\beta \mid \beta \in D\}$ . Hence it becomes an automorphism group of  $(M_D, Y)$ . ■

**Definition 9** *For a word  $\beta$ ,  $M_D$  has an automorphism  $\delta_\beta$  given by*

$$\delta_\beta : u^\alpha \rightarrow (-1)^{\langle \beta, \alpha \rangle} u^\alpha \quad (4.36)$$

for  $u^\alpha \in M_\alpha$ . This is an automorphism induced from the inner automorphism of  $M = M_{\bar{0}} \oplus M_{\bar{1}}$  and we will call this an inner automorphism of  $M_D$ .

**Notation** We always fix a lowest weight vector  $q$  of  $M_{\bar{1}} = L(\frac{1}{2}, \frac{1}{2})$  with  $q_{-2}q = 2\mathbf{w}$ , see (4.9) and (4.10) and  $q^i$  denotes such  $q$  of each  $M^i$ . For a word  $\alpha = (a^1, \dots, a^n)$  of length  $n$ , set

$$q^\alpha = (\otimes_{i=1}^n q^{a^i}), \quad (4.33)$$

where  $q^{a^i} = q^i$  if  $a_i = 1$  and  $q^{a^i} = \mathbf{1}^i$  if  $a_i = 0$ . Then  $q^\alpha \otimes e^\alpha$  is a lowest weight vector of  $M_\alpha$ . For a  $T$ -module  $X = \otimes_{i=1}^n L(\frac{1}{2}, b^i) \otimes R$  with some formal vector space  $R$ , we define an

intertwining operator  $I(q^\alpha \otimes e^\alpha, z)$  of  $q^\alpha \otimes e^\alpha$  of type  $\begin{pmatrix} \otimes L(\frac{1}{2}, h^i + a^i/2) \otimes e^\alpha R \\ \otimes L(\frac{1}{2}, a^i/2) \otimes e^\alpha & \otimes L(\frac{1}{2}, h^i) \otimes R \end{pmatrix}$  by

$$(\otimes I)(q^\alpha \otimes e^\alpha, z) = (\otimes I^{(h^i)})(q^\alpha \otimes e^\alpha, z) = \otimes_{i=1}^n (I^{a^i/2, h^i}(q^{a^i}, z)) \otimes e^\alpha, \quad (4.34)$$

where  $\alpha = (a^i)$  and  $I^{1/2, h^i}(q^i, z)$  as in (4.12) and  $I^{0, h^i}(\mathbf{1}^i, z) = 1$ . Here  $h^i + a^i/2$  denotes the fusion rules (1.1), that is,  $0+0 = \frac{1}{2} + \frac{1}{2} = 0$ ,  $0 + \frac{1}{2} = \frac{1}{2} + 0 = \frac{1}{2}$ ,  $\frac{1}{16} + 0 = \frac{1}{16} + \frac{1}{2} = \frac{1}{16}$ . By Remark 4, the powers of  $z$  in  $I^{\frac{1}{2}, \frac{1}{16}}(u, z)$  are  $\frac{1}{2} + \mathbf{Z}$  for  $u \in M_{\bar{1}}$  and those of  $z$  in the other intertwining operators are all integers. Hence, for an even word  $\alpha = (a^i)$  and a  $T$ -module  $X$  with the  $\frac{1}{16}$ -word  $\tilde{h}(X)$ , the powers of  $z$  in  $(\otimes I)(q^\alpha \otimes e^\alpha, z)$  are in  $\mathbf{Z} + \frac{1}{2}\langle \alpha, \tilde{h}(X) \rangle$ , that is,

$$(\otimes I)(q^\alpha \otimes e^\alpha, z) \in \text{Hom}(X, M_\alpha \times X)[[z, z^{-1}]]z^{\frac{1}{2}\langle \alpha, \tilde{h}(X) \rangle}. \quad (4.35)$$

We have thought of even code. We will next treat words of odd weights, too. By the direct calculation, we have the following.

**Lemma 4.5** *For  $u \in M_\delta$ ,  $v \in M_\gamma$ ,  $Y(u, z_1)Y(v, z_2) \sim (-1)^{|\delta||\gamma|}Y(v, z_2)Y(u, z_1)$ , where  $|\delta|$  and  $|\gamma|$  denote the weights of  $\delta$  and  $\gamma$ , respectively.*

Namely, we have the following result:

**Theorem 4.4**  $(M_{\mathbf{Z}_2^n}, Y, \mathbf{w}, \mathbf{1})$  is a SVOA.

**Remark 6** *If  $D$  is an even linear code of length  $n$ , then  $M_D$  is a vertex operator subalgebra of  $M_{\mathbf{Z}_2^n}$ . As  $M_D$ -modules,  $M_{\mathbf{Z}_2^n}$  splits into a direct sum  $\oplus_{\alpha \in \mathbf{Z}_2^n/D} M_{\alpha+D}$  of irreducible  $M_D$ -submodules  $M_{\alpha+D} = \sum_{\beta \in \alpha+D} M_\beta$ .*

**Definition 10** *Since  $M_0^i \cong L(\frac{1}{2}, 0)$  contains a conformal vector  $\mathbf{w}^i$  with central charge  $\frac{1}{2}$  as the Virasoro element,  $M_{(0, \dots, 0)} = \otimes_{i=1}^n L(\frac{1}{2}, 0)$  contains mutually orthogonal  $n$  conformal vectors*

$$e^i = \mathbf{1}^1 \otimes \dots \otimes \mathbf{1}^{i-1} \otimes \mathbf{w}^i \otimes \mathbf{1}^{i+1} \otimes \dots \otimes \mathbf{1}^n. \quad (4.37)$$

*We will call the set  $\{e^1, \dots, e^n\}$  “the coordinate conformal vectors”.*

The most useful example in this paper is a VOA  $M_{H_8}$  constructed from the  $[8, 4, 4]$ -Hamming code  $H_8$ , which we will denote by  $V_{H_8}$  and we will classify its representations in Sec.5.

### 4.3 Uniqueness of tensor product construction

Let  $(V, Y)$  be a simple VOA with a set  $\{e^i \mid i = 1, \dots, n\}$  of mutually orthogonal rational conformal vectors such that the sum  $\sum_{i=1}^n e^i$  is the Virasoro element  $\mathbf{w}$ . Set

$$T = \langle e^1, \dots, e^n \rangle \cong \otimes_{i=1}^n L(\frac{1}{2}, 0).$$

Assume that there is no  $\frac{1}{16}$  entry in the set  $\text{lwr}(T, V)$  of lowest weight rows  $(h^1, \dots, h^n)$  of  $V$ .

We will prove the following result:

**Theorem 4.5** *Under the above assumptions,  $(V, Y)$  is isomorphic to  $M_D$  for some even linear binary code  $D$ .*

[**Proof**] By our assumption, every  $e^i$  is a conformal vector of type two and defines an automorphism:

$$\sigma_{e_i} : \begin{cases} 1 & \text{on } U \cong L(\frac{1}{2}, 0) \quad \text{as } \langle e_i \rangle\text{-modules,} \\ -1 & \text{on } U \cong L(\frac{1}{2}, \frac{1}{2}) \quad \text{as } \langle e_i \rangle\text{-modules,} \end{cases} \quad (4.38)$$

by the second binary mode in (1.2), see [M1]. Set  $R = \langle \sigma_{e_i} \mid i = 1, \dots, n \rangle$ . Then  $R$  is an elementary Abelian 2-group. Let  $\text{Irr}(R)$  denote the set of all irreducible linear characters of  $R$ . Then we have the decomposition  $V = \bigoplus_{\chi \in \text{Irr}(R)} V^\chi$  into the direct sum of irreducible  $V^R$ -modules  $V^\chi$ , where  $V^\chi = \{v \in V \mid g(v) = \chi(g)v \text{ for all } g \in R\}$  is the weight space of  $\chi$ . Then the fixed point space  $V^R$  is a direct sum of irreducible  $T$ -modules isomorphic to  $\otimes L(\frac{1}{2}, 0)$ . Since  $\dim V_0 = 1$ , we have  $V^R \cong \otimes L(\frac{1}{2}, 0)$  and so  $V^R = T$ . By the result in [DM],  $V^\chi$  is an irreducible  $T$ -module and hence we have  $V^\chi \cong M_{d(\chi)}$  as  $T$ -modules for some binary word  $d(\chi)$ . Let  $D$  to be the set of all such words  $d(\chi)$ . As they also proved by Lemma 3.1 in [DM], for two  $\chi, \phi \in \text{Irr}(R)$ , there are  $u \in V_\chi, v \in V_\phi$  and  $n \in \mathbf{Z}$  such that  $u_n v \neq 0$ . Hence, by the fusion rules for Ising models (1.1), we have that  $D$  is closed under the addition, that is,  $D$  is a binary linear code. Since the weight of elements in  $V$  are all integers,  $D$  is an even code and  $V \cong M_D$  as  $T$ -modules. Let us recall the fusion rules of irreducible modules of Ising model:

$$L(\frac{1}{2}, 0) \text{ is an identity and } L(\frac{1}{2}, \frac{1}{2}) \times L(\frac{1}{2}, \frac{1}{2}) = L(\frac{1}{2}, 0).$$

These imply that the intertwining spaces  $I \begin{pmatrix} 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$  and  $I \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$  have one dimension. Choose a fixed vector  $q$  of  $(M_{\overline{1}})_{\frac{1}{2}}$  so that  $q_{-2}q = 2\mathbf{w}$ , where  $\mathbf{w}$  is the Virasoro element



of  $M_0^- = L(\frac{1}{2}, 0)$ . Let  $I^{\frac{1}{2}, 0}(q, z)$  and  $I^{\frac{1}{2}, \frac{1}{2}}(q, z)$  be intertwining operators of  $q$  of types  $\begin{pmatrix} L(\frac{1}{2}, 0) \\ L(\frac{1}{2}, \frac{1}{2}) & L(\frac{1}{2}, \frac{1}{2}) \end{pmatrix}$  and  $\begin{pmatrix} L(\frac{1}{2}, \frac{1}{2}) \\ L(\frac{1}{2}, \frac{1}{2}) & L(\frac{1}{2}, 0) \end{pmatrix}$ , respectively, such that

$$Y(q, z) = \begin{pmatrix} 0 & I^{\frac{1}{2}, \frac{1}{2}}(q, z) \\ I^{\frac{1}{2}, 0}(q, z) & 0 \end{pmatrix} \in \text{End}(M)[[z, z^{-1}]] \quad (4.39)$$

is the vertex operator of  $q$  in the SVOA  $M = M^0 \oplus M^1$  as in (4.9). For  $\beta = (b^1, \dots, b^n) \in D$ , take a lowest weight vector  $q^\beta \otimes e^\beta = (\otimes_i q^{b^i}) \otimes e^\beta$  of  $M_\beta$  as in (4.33). Since the intertwining space has one dimensional, the restriction:

$$Y(q^\beta \otimes e^\beta, z)|_{M_\xi} : M_\xi \rightarrow M_{\xi+\beta} \quad (4.40)$$

is a scalar times of  $(I^\xi)(q^\beta \otimes e^\beta, z) = (\otimes_i I^{b^i/2, *}(q^{b^i}, z)) \otimes e^\beta$  for  $\beta, \xi \in D$ , say

$$Y(q^\beta \otimes e^\beta, z)|_{M_\xi} = \lambda(\beta, \xi) I^\xi(q^\beta \otimes e^\beta, z).$$

Since this is true for any other elements, we have

$$Y(u^\beta \otimes e^\beta, z)|_{M_\xi} = \lambda(\beta, \xi) I^\xi(u^\beta \otimes e^\beta, z) \quad (4.41)$$

for any  $u^\beta \otimes e^\beta \in M_\beta$ . The mutual commutativity of  $\{Y(u^\beta \otimes e^\beta, z) \mid \beta \in D\}$  imply

$$\lambda(\beta, \alpha + \xi) \lambda(\alpha, \xi) = \lambda(\alpha, \beta + \xi) \lambda(\beta, \xi) \quad (4.42)$$

for any  $\alpha, \beta, \xi \in D$ . Namely,  $\lambda(*, *)$  is a 2-cocycle. Since  $Y$  and  $I^\xi$  satisfy the associativity, we have

$$\lambda(\beta, \alpha + \xi) \lambda(\alpha, \xi) = \lambda(\alpha, \beta + \xi) \lambda(\beta, \xi) = \lambda(\alpha + \beta, \xi). \quad (4.43)$$

In particular,  $\lambda(0, \beta) = 1$  for any  $\beta$  and  $\lambda(\alpha, \beta) = \pm 1$ . Using the skew-symmetry, we have  $\lambda(\beta, 0) = 1$  for any  $\beta$ . Substituting  $\xi = \alpha$  into (4.43), we have  $\lambda(\alpha, \alpha) = \lambda(\alpha + \beta, \alpha)$  for any  $\alpha, \beta$ . Hence  $\lambda(\alpha, \beta) = \lambda(\beta, \beta) = \lambda(0, \beta) = 1$  for any  $\alpha, \beta$ . Namely,  $Y$  is uniquely determined. ■

## 4.4 Examples

Let  $L_0 = \mathbf{Z}x^1 + \mathbf{Z}x^2 + \dots + \mathbf{Z}x^n$  be a lattice with  $\langle x^i, x^j \rangle = \delta_{i,j}$ . Then  $L = \{\sum a^i x^i : \sum a^i \equiv 2 \pmod{2}\}$  is an even lattice. Let  $V_L$  is the lattice VOA constructed by [FLM].  $V_L$  contains  $V_{2\mathbf{Z}x_1} \otimes \dots \otimes V_{2\mathbf{Z}x_n}$  and hence  $V_L$  contains a set  $\{e^1, e^2, \dots, e^{2n-1}, e^{2n}\}$  of mutually orthogonal  $2n$  conformal vectors. Set  $T = \langle e^1, \dots, e^{2n} \rangle$ . Since  $V_L$  is a subspace of  $V_{\mathbf{Z}x^1} \otimes \dots \otimes V_{\mathbf{Z}x^n}$ , there is no  $\frac{1}{16}$ -entries in  $\text{lwr}(T, V_L)$ . Let  $D$  be the code consisting of all even words of length  $2n$ . It is easy to see that  $V_L$  is isomorphic to  $M_D$ .

## 5 $M_D$ -modules

### 5.1 Classification

In this subsection, we will classify the irreducible  $M_D$ -modules and study their fusion rules.

Let  $\{e^1, \dots, e^n\}$  be the set of coordinate conformal vectors of  $M_D$  and set  $T = \langle e^1 \rangle \otimes \dots \otimes \langle e^n \rangle \cong \otimes_{i=1}^n L(\frac{1}{2}, 0)$ . We note that all irreducible  $T$ -submodules of  $M_D$  has no  $L(\frac{1}{2}, \frac{1}{16})$ -entry. Let  $(X, Y^X)$  be an irreducible  $M_D$ -module. Since  $T$  is rational by Corollary 3.1, we have a decomposition  $X = \oplus U^i$  of  $X$  as a direct sum of irreducible  $T$ -submodules  $U^i$ . By the fusion rules for Ising model, the  $\frac{1}{16}$ -words of any irreducible  $T$ -submodules of  $X$  are all the same, say  $\tilde{h}(X)$ . By Remark 4.(1), we have:

**Lemma 5.1**  $\tilde{h}(X)$  is orthogonal to  $D$ .

Set  $K = \{\alpha \in D \mid \alpha \subseteq \tilde{h}(X)\}$  and let  $H$  be a maximal self-orthogonal subcode of  $K$ . Since  $K$  is an even code, we have  $H^\perp \cap K \subseteq H$ . Let  $\hat{K} = \{\pm e^k : k \in K\}$  denote the central extension group of  $K$  by  $\pm 1$  induced from the inner products as in (4.26) and (4.27), then  $\{\pm e^\alpha : \alpha \in H\}$  is a maximal normal Abelian subgroup of  $\hat{K}$ . Our aim in this subsection is to show that every irreducible  $M_H$ -submodule of  $X$  determines the  $M_D$ -module structure on  $X$  uniquely.

**Theorem 5.1** Let  $(X, Y^X)$  be an irreducible  $M_D$ -module and  $\{X^i : i = 1, \dots, k\}$  the set of non-isomorphic irreducible  $T$ -submodules of  $X$ . Then there are irreducible  $\hat{K}$ -modules  $Q^i$  on which  $-e^0 \in \hat{K}$  acts as  $-1$  such that  $X \cong \oplus_{i=1}^k (Q^i \otimes X^i)$  as  $M_K$ -modules.

**[Proof]** Let  $U$  be a homogeneous component of  $X$  generated by all  $T$ -submodules isomorphic to  $X^1$  and  $U = \oplus_{i=1}^k U^i$  the decomposition of  $U$  into a direct sum of irreducible  $T$ -submodules  $U^i$ . For  $\alpha = (a^i) \in K$ , set  $q^\alpha$  as in (4.33). By the fusion rules for Ising module (1.1),  $Y^X(q^\alpha \otimes e^\alpha, z)U \subset U[[z, z^{-1}]]$  for  $\alpha \in K$ . Since the both of the tensor products of  $L(\frac{1}{2}, 0)$  and  $L(\frac{1}{2}, \frac{1}{2})$  with any irreducible  $L(\frac{1}{2}, 0)$ -modules are irreducible, the vertex operator  $Y^X(u^\alpha \otimes e^\alpha, z)|_U$  of  $u^\alpha \otimes e^\alpha$  has an expression

$$A(e^\alpha) \otimes ((\otimes I)(u^\alpha, z)),$$

for  $u^\alpha \otimes e^\alpha \in M_\alpha$ . where  $A(e^\alpha)$  is a  $k \times k$ -matrix and  $(\otimes I)(*, z)$  is the tensor product of the intertwining operators in (4.12) and (4.13). Since  $Y^X(u^\alpha, z)$  satisfies the commutativity

and  $(\otimes I)(u^\alpha, z)$  satisfies the super-commutativity, we see the supercommutativity :

$$A(\alpha)A(\beta) = (-1)^{(\alpha, \beta)} A(\beta)A(\alpha).$$

Furthermore, since the both  $Y^X(*, z)$  satisfies the associativity and  $(\otimes I)(u^\alpha, z)$  satisfy the superassociativity, we see the associativity :

$$A(e^\alpha)A(e^\beta) = A(e^\alpha e^\beta)$$

and  $A(\alpha)A(\alpha)$  is the identity matrix for all  $\alpha, \beta \in K$ . This implies that  $A$  is a matrix representation of the central extension  $\hat{K}$  and  $U = Q^1 \otimes X^1$  for a  $\hat{K}$ -module  $Q^1$ . Suppose that  $Q^1$  is not an irreducible  $\hat{K}$ -module and  $Q^0$  is a proper submodule. Let  $W$  be a subspace spanned by  $\{v_n w : v \in M_D, w \in Q^0, n \in \mathbf{Z}\}$ . By Proposition 4.1 in [DM], we have  $X = W$ . On the other hand, by the fusion rules for Ising model (1.1),  $M_{\beta+K} \times Q^0 \otimes X^1$  does not contains a submodule isomorphic to  $X^1$  for  $\beta \notin K$  and so  $W \cap U = Q^0 \otimes X^1$ , a contradiction. Hence,  $Q^1$  is an irreducible  $\hat{K}$ -module on which  $-e^0$  acts as  $-1$ . ■

As a corollary, we have the following:

**Corollary 5.1** *If  $E$  is a self-orthogonal binary linear code and  $X$  is an irreducible  $M_E$ -module, then the multiplicities of irreducible  $T$ -submodules of  $X$  are at most one. In particular, if  $X$  contains a  $T$ -submodule  $\otimes_{i=1}^n L(\frac{1}{2}, \frac{1}{16})$ , then  $X \cong \otimes_{i=1}^n L(\frac{1}{2}, \frac{1}{16})$  as  $T$ -modules.*

We next prove that  $M_D$  is rational by the same argument. Namely,  $M_D$  has finitely many non-isomorphic irreducible modules and every module is a direct sum of irreducible modules.

**Proposition 5.1**  *$M_D$  is rational.*

[**Proof**] Let  $W$  be an indecomposable  $M_D$ -module. We will show that  $W$  is an irreducible  $M_D$ -module. Suppose false and let  $U$  be a proper  $M_D$ -submodule of  $W$ . Viewing  $W$  as a  $T$ -module,  $W$  is a direct sum  $W = \oplus W^i$  of irreducible  $T$ -modules and assume  $W^1 \subseteq U$ . Since the action of  $M_D$  does not change the  $\frac{1}{16}$ -words, every  $W^i$  has the same  $\frac{1}{16}$ -word  $\beta$  and the set  $\text{lwr}(T, W)$  of lowest weight rows is an orbit of  $D$ . Set  $K = \{\alpha \in D : \alpha \subseteq \beta\}$  and let  $S$  be a direct sum of all  $W^i \cong W^1$ . By the above theorem,  $S \cong Q \otimes W^1$  and  $S \cap U \cong Q^0 \otimes W^1$  for a  $\hat{K}$ -module  $Q$  and its submodule  $Q^0$ . Since the  $\mathbf{C}\hat{K}$ -modules are completely reducible,  $Q$  is a direct sum  $Q^0 \oplus Q^1$  for some  $\hat{K}$ -module  $Q^1$ . By the assumption, we have  $Q^1 \neq 0$ . Let  $U^i$  be the space spanned by

$\{v_n w : v \in M_D, w \in Q^i \otimes W^1, n \in \mathbf{Z}\}$  for  $i = 0, 1$ . By Proposition 4.1 in [DM], they are  $M_D$ -modules. Then by the fusion rules,  $U^0 \cap U^1 \cap Q \otimes W^1 = 0$ . Namely,  $U^0 \cap U^1$  does not contain  $T$ -submodules isomorphic to  $W^1$ . Since the set  $lwr(T, W)$  is an orbit of  $D$ , we obtain  $U^0 \cap U^1 = 0$  and so  $W = U^0 \oplus U^1$ , which contradicts to the assumption. ■

## 5.2 Induced module

We next prove that  $M_D$ -modules containing a fixed irreducible  $M_K$ -module are uniquely determined. Let's construct such an  $M_D$ -module.

Since  $\hat{K}$  is a direct product of an Abelian group and an extra-special 2-group, it is known that every irreducible  $\hat{K}$  representation  $\phi$  with  $\phi(-e^0) = -1$  is induced from a linear representation  $\chi$  of a maximal Abelian normal subgroup  $\hat{H}$ .

Let  $\beta \in D^\perp$  and  $K = \{\alpha = (a^i) \in D \mid \alpha \subseteq \beta\}$ . Let  $H$  be a self-orthogonal subcode of  $K_\beta$  and choose any linear irreducible character  $\chi : \pm \hat{H} \rightarrow \pm 1$  with  $\chi(-e^0) = -1$ . Let  $F_\chi$  be a one dimensional  $\hat{H}$ -module such that  $e^\alpha p = \chi(e^\alpha) p$  for  $p \in F_\chi, \alpha \in H$ . Take any  $h^i = 0, \frac{1}{2}, \frac{1}{16}$  such that  $\frac{1}{16}$ -word of  $(h^i)$  is  $\beta$ . Then  $U = \otimes L(\frac{1}{2}, h^i) \otimes F_\chi$  is an  $T$ -module. Define the module vertex operator  $Y^U((\otimes u^i) \otimes e^\alpha, z)$  for  $u^i \in M\overline{a^i}$  on  $U$  and  $(a^i) \in H$  by

$$(\otimes_{i=1}^n I^{a^i/2, h^i}(u^i, z)) \otimes \chi(e^\alpha) \quad (5.1)$$

and extend it to all elements in  $M_H$  linearly. By Remark 5 and the associativity (4.19), we have:

**Lemma 5.2**  *$Y^U(v, z)$  are all well defined for  $v \in M_H$  and they satisfies the derivation, the associativity and the mutually commutativity. In particular,  $U$  is an  $M_H$ -module.*

We denote the above  $M_H$ -module by  $U((h^i)) \otimes F$ . Let  $D$  be an even linear binary code and assume that  $D$  is orthogonal to  $\tilde{h}(U)$ . We will define an induced  $M_D$ -module  $X = \text{Ind}_{M_H}^{M_D}(U(h^i), \chi)$  from  $U((h^i), \chi)$  as follows:

Let  $\{\beta^1 = (b_i^1), \dots, \beta^s = (b_i^s)\}$  be a transversal of  $H$  in  $D$ , then  $\{e^{\beta^i} : i = 1, \dots, s\}$  is a transversal of  $\hat{H}$  in  $\hat{D}$ . Set

$$X = \oplus_{(\beta^i) \in D/H} \{U(h^i + \frac{b^i}{2}) \otimes (e^{\beta^i} \otimes_{\hat{H}} F_\chi)\}, \quad (5.2)$$

where  $h^i + \frac{b^i}{2}$  denotes the fusion rules for Ising models. Since  $e^{\beta+h} \otimes_{\hat{H}} F_\chi = e^\beta \otimes F_\chi$  for  $h \in H$  and  $(h^i + \frac{b^i}{2}) = (h^i)$  for  $(b^i) \in H$ ,  $X$  does not depend on the choice of transversal of  $H$  in  $D$  and  $X$  becomes an  $M_H$ -module.

We define the module vertex operator  $Y(u^\gamma \otimes e^\gamma, z)$  of  $u^\gamma \otimes e^\gamma$  by  $(\otimes I)(u^\gamma, z)$  on the first term and  $e^\gamma$  on the second term for  $\gamma \in D$ , where  $(\otimes I)(u^\gamma, z)$  is as in (4.34). We denote this module by  $Ind_{M_H}^{M_D}(U)$ . We will show that these actions are all well defined and they satisfy the commutativity, the derivation and the associativity. Namely, we will prove the following:

**Proposition 5.2**  $Ind_{M_H}^{M_D}(U)$  is an  $M_D$ -module.

[**Proof**] First, we assume  $\tilde{h}(U) = (1^n)$ . Then,  $X = \oplus U((\frac{1}{16})_i) \otimes Q$ , where  $Q$  is a  $\hat{K}$ -module induced from  $F$ . Since  $(\otimes I)(u^\alpha, z)$  satisfies the derivation, the supercommutativity (4.16) and the superassociativity (4.19),  $Y(u^\alpha \otimes e^\alpha, z)$  satisfies the derivation, the commutativity and the associativity. Hence,  $X$  is an  $M_D$ -module. We next assume that  $\tilde{h}(U) = (0^n)$ . Then,  $H = \{(0^n)\}$  and  $F = F_\chi$  is a trivial module and  $X = \oplus U(h^i + b^i/2) \otimes e^{\beta^i}$  is isomorphic to  $M_{(h^i)+D}$ . Hence,  $X$  is also an  $M_D$ -module. We next treat the general case. By the permutation of coordinates, we may assume  $\tilde{h}(U) = (0^s 1^t)$ . Let  $S_n$  be the set of all even words of length  $n$ . Since  $D$  is orthogonal to  $(0^s 1^t)$ ,  $D$  is a subcode of  $S_s \oplus S_t$ . Divide the coordinates into the first  $s$  coordinates and the last  $t$  coordinates and set  $H = H^0 \oplus H^1$ , where  $H^0 = \{(0^s)\}$  and  $H^1 = \{(a^{s+1}, \dots, a^{s+t}) : (a^i) \in H\}$ . Clearly,  $U \cong T^s \otimes U^1$  as  $M_{H^0} \otimes M_{H^1}$ -modules, where  $T^s \cong \otimes_{i=1}^s L(\frac{1}{2}, h^i)$  is an  $M_{H^0}$ -module and  $U^1 \cong \otimes_{i=s+1}^{s+t} L(\frac{1}{2}, \frac{1}{16})$  is an  $M_{H^1}$ -module. Clearly, we see

$$Ind_{M_H}^{M_D}(U) \subseteq Ind_{M_H}^{M_{S_s \oplus S_t}}(U) \subseteq Ind_{M_{H^0}}^{M_{S_s}}(T^s) \otimes Ind_{M_{H^1}}^{M_{S_t}}(U^1).$$

As we showed,  $Ind_{M_{H^0}}^{M_{S_s}}(T^s)$  is an  $M_{S_s}$ -module and  $Ind_{M_{H^1}}^{M_{S_t}}(U^1)$  is an  $M_{S_t}$ -module. We hence have an  $M_{S_s \oplus S_t}$ -module  $Ind_{M_H}^{M_{S_s \oplus S_t}}(U)$ , which is also an  $M_D$ -module containing  $Ind_{M_H}^{M_D}(U)$ . Therefore, the module vertex operators satisfy the commutativity, the derivations, and the associativity. Since  $Ind_{M_H}^{M_D}(U)$  is invariant under the actions of  $M_D$ , we have the desired results. ■

Since  $q^\gamma$  for  $\gamma \in D - H$  acts regularly on the set of irreducible  $T$ -modules  $U$  with a  $\frac{1}{16}$ -word  $\beta$ , we have that if  $M_D$ -module  $X$  contains a  $M_H$ -module  $U$ , then  $X$  has to have the above structure on it. Therefore, we have proved the following theorem.

**Theorem 5.2** Let  $X$  be an irreducible  $M_D$ -module with a  $\frac{1}{16}$ -word  $\beta$ . Set  $K = \{\alpha \in D | \alpha \subseteq \beta\}$  and let  $H$  be a maximal self-orthogonal subcode of  $K$ . Then there is a pair  $((h^i), \chi)$  of a lowest weight row  $h = (h^1, \dots, h^n)$  and a linear character  $\chi$  of  $H$  such that  $X = Ind_{M_H}^{M_D}(U((h^i), \chi))$ .

As corollaries of the above theorem, we have the following results:

**Theorem 5.3** *Let  $X$  be an  $M_D$ -module with a  $\frac{1}{16}$ -word  $\tilde{h}$ . Set  $K = \{\alpha \in D : \alpha \subseteq \tilde{h}\}$  and let  $H$  be a maximal self-orthogonal subcode of  $K$ . Then the  $M_D$ -module structure on  $X$  is uniquely determined by an  $M_H$ -submodule of  $X$ .*

**Corollary 5.2** *Let  $D$  be an even binary linear code containing a self-dual subcode  $E$ . Assume that  $X$  is an irreducible  $M_D$ -module with a  $\frac{1}{16}$ -word  $\beta$ . Set  $K = \{\alpha \in D | \alpha \subseteq \beta\}$  and  $H = \{\alpha \in K | \langle \alpha, \beta \rangle = 0 \text{ for all } \beta \in K \cap E\}$ , then  $X = \sum M^i \otimes T^i$  as  $M_E$ -modules, where  $M^i$  are non-isomorphic  $M_E$ -modules and  $T^i$  are irreducible  $\hat{H}$ -modules. In particular, the multiplicity of  $M^i$  in  $X$  is  $\sqrt{|E + H : E|}$ . We note that the actions of  $M_E$  and  $M_H$  are not always commutative.*

**[Proof]** Let  $H'$  be a maximal self-orthogonal subcode of  $K$  containing  $K \cap E$ . By the above theorem, there is an irreducible  $M_{H'}$ -module  $U = \otimes L(\frac{1}{2}, h^i)$  such that  $X = \text{Ind}_{M_{H'}}^{M_D}(U)$ . Clearly,  $U$  is irreducible as  $T$ -module and  $T \subseteq M_E$ . Let  $M^1$  be a irreducible  $M_E$ -submodule of  $X$  containing  $U$ . Let  $M$  denote the  $M_E$ -submodule generated by all irreducible  $M_E$ -submodule isomorphic to  $M^1$ . Then by Theorem 5.2, we obtain

$$M^1 = \sum_{\beta^j \in E/H \cap E} U(h^i + b_i^j) \otimes e^{\beta^j} \otimes F.$$

Clearly, as  $T$ -submodules, we have

$$\begin{aligned} M &\subseteq \sum_{\beta^j \in (E+K)/H'} U(h^i + b_i^j) \otimes e^{\beta^j} \otimes F \\ &= \sum_{\gamma^j \in (K+E)/(E+H')} (\sum_{\beta^k \in (E+H')/H'} U(h^i + b_i^j + \gamma_i^k) \otimes e^{\beta^k}) e^{\gamma^j} \otimes F \\ &= \sum_{\gamma^j \in (K+E)/(E+H')} (\sum_{\beta^k \in E/(E \cap H')} U(h^i + b_i^j) \otimes e^{\beta^k}) e^{\gamma^j} \otimes F \\ &= \sum_{\gamma^j \in (K+E)/(E+H')} M^1 e^{\gamma^j} \otimes F. \end{aligned}$$

It follows from direct calculation that  $M^1 e^{\gamma^j} \otimes F$  is isomorphic to  $M^1$  if and only if  $\gamma^j \in K$  and  $\langle \gamma^j, \beta \rangle = 0$  for all  $\beta \in E \cap K$ . We hence have the desired decomposition and multiplicity. ■

### 5.3 Fusion rule of $M_D$ -modules

Let us calculate the fusion rule of  $M_D$ -modules in this subsection. Restricting Proposition 11.9 in [DL] into our case, we have:

**Theorem 5.4** *Let  $E$  be a subcode of  $D$ . Let  $W^1, W^2, W^3$  be irreducible  $M_D$ -module and  $U^1, U^2$  irreducible  $M_E$ -submodules of  $W^1$  and  $W^2$ , respectively, then there is an injection map:*

$$\phi : I_{M_D} \begin{pmatrix} W^3 \\ W^1 & W^2 \end{pmatrix} \rightarrow I_{M_E} \begin{pmatrix} W^3 \\ U^1 & U^2 \end{pmatrix}.$$

[**Proof**] For  $I(*, z) \in I_{M_D} \begin{pmatrix} W^3 \\ W^1 & W^2 \end{pmatrix}$  and  $v \in U^1$ ,  $I'(v, z) = I(v, z)|_{U^2} \in \text{Hom}(U^2, W^3)\{z, z^{-1}\}$ . Set  $\phi(I(*, z)) = I'(*, z)$ . Clearly,  $I'(*, z) \in I_{M_E} \begin{pmatrix} W^3 \\ U^2 & U^2 \end{pmatrix}$  and hence we have a map

$$\phi : I_{M_D} \begin{pmatrix} W^3 \\ W^1 & W^2 \end{pmatrix} \rightarrow I_{M_E} \begin{pmatrix} W^3 \\ U^1 & U^2 \end{pmatrix}.$$

Clearly,  $\phi$  is a linear map. We will show that  $\phi$  is injective. Namely, we will prove that if  $I(v, z)|_{U^2} = 0$  for all  $v \in U^1$ , then  $I(*, z) = 0$ . By the commutativity of intertwining operators, we see  $0 = (z_1 - z_2)^N Y^3(u, z_1) I(v, z_2) w = (z_1 - z_2)^N I(v, z_2) Y^2(u, z_1) w$  for all  $u \in M_D$  and  $w \in U^2$ . Suppose  $I(v, z_2) Y^2(u, z_1) w \neq 0$ , there is an integer  $r$  such that  $I(v, z_2) u_{r-1}^2 w \neq 0$  and  $I(v, z_2) Y(u, z_1) w = I(v, z_2) (\sum_{i=0}^r (u_{-i+r-1}^2 w) z_1^{i-r})$ . However, since  $(z_1 - z_2)^N I(v, z_2) z_1^r Y(u, z_1) w = 0$  for a sufficiently large integer  $N$ , we obtain  $(-z_2)^N I(v, z_2) u_{r-1} w = 0$  by substituting 0 into  $z_1$  and so we have  $I(v, z_2) u_{r-1} w = 0$ , which contradicts to the choice of  $r$ . Hence,  $I(v, z_2) Y^2(u, z_1) w = 0$ . Since the set  $\{u_m w : m \in \mathbf{Z}, u \in M_D, w \in U^2\}$  spans  $W^2$  by Proposition 4.1 in [DM], we have  $I(v, z_2) w = 0$  for all  $w \in W^2$ . For  $u \in M_D, v \in U^1, w \in W^2$  and  $n \in \mathbf{Z}$ ,

$$I(u_n v, z) u = \text{Res}_{z_1} \{(z_1 - z)^n Y^3(u, z_1) I(v, z) w - (-z + z_1)^n I(v, z) Y^2(u, z_1) w\} = 0.$$

Hence, it follows from the associativity of intertwining operators that  $I(v, z) = 0$  for all  $v \in W^1$ . ■

As a corollary of the above theorem, we will prove the following fusion rule.

**Theorem 5.5** *Let  $X$  be an irreducible  $M_D$ -module with a  $\frac{1}{16}$ -word  $\tilde{h}(X)$ , then the tensor product*

$$M_{\alpha+D} \times X$$

*is an irreducible  $M_D$ -module. Furthermore, if  $\tilde{h}(X)$  is orthogonal to  $\alpha$ , then the intertwining operators  $I(v, z)$  of type  $\begin{pmatrix} M_{\alpha+D} \times X \\ M_{\alpha+D} & X \end{pmatrix}$  are in  $\text{Hom}(X, M_{\alpha+D} \times X)[[z, z^{-1}]]$  for all  $v \in M_{\alpha+D}$ . Namely, the powers  $z$  in  $I(v, z)$  are all integers. If  $\tilde{h}(X)$  is not orthogonal to  $\alpha$ , then  $I(v, z) \in \text{Hom}(X, M_{\alpha+D} \times X)[[z, z^{-1}]] z^{\frac{1}{2}}$ .*

**[Proof]** Set  $\alpha = (a^i)$  and  $K = \{\beta \in D | \beta \subseteq \tilde{h}(X)\}$ . Let  $U$  be an irreducible  $M_D$ -module such that  $I_{M_D} \begin{pmatrix} U \\ M_{\alpha+D} & X \end{pmatrix} \neq 0$ . By Theorem 5.1, we have a decomposition  $X = \oplus_{\beta=(b^i) \in D/K} (X^\beta \otimes Q^\beta)$  as a direct sum of irreducible  $M_K$ -modules  $Q^\beta \otimes X^\beta$ , where  $Q^\beta$  are irreducible  $\hat{K}$ -modules and  $X^\beta = \otimes L(\frac{1}{2}, h^i + b^i/2)$  are irreducible  $T$ -modules for some  $h = (h^i)$ . Similarly, we have  $U = \oplus (Q^\gamma \otimes U^\gamma)$  with irreducible  $\hat{K}$ -modules  $Q^\gamma$  and irreducible  $T$ -modules  $U^\gamma = \otimes L(\frac{1}{2}, h^i + b^i/2 + a^i/2)$ . By Theorem 5.4, we obtain

$$\dim I_{M_D} \begin{pmatrix} U \\ M_{\alpha+D} & X \end{pmatrix} \leq \dim I_{M_K} \begin{pmatrix} Q^\beta \otimes X^\beta \\ M_{\alpha+K} & Q^\gamma \otimes U^\gamma \end{pmatrix},$$

where  $\gamma = \alpha + \beta$ . For  $q^\alpha \otimes e^\alpha \in M_{\alpha+K}$ , the intertwining operator  $I(q^\alpha \otimes e^\alpha, z)$  is expressed by

$$A \otimes ((\otimes I)(q^\alpha, z) \otimes e^\alpha)$$

for some  $k \times k$ -matrix  $A$ . Moreover, the powers of  $z$  in  $I(q^\alpha \otimes e^\alpha, z)$  are in  $\mathbf{Z} + \frac{1}{2}\langle \tilde{h}, \alpha \rangle$  by (4.35). It follows from the commutativity of intertwining operators that  $A$  satisfies the relation

$$A\phi(s) = \psi(s)A \tag{5.4}$$

for all  $s \in \hat{K}$ , where  $\phi$  and  $\psi$  are the representations of  $\hat{K}$  on  $Q^\beta$  and  $Q^\gamma$ , respectively. Since  $\phi$  and  $\psi$  are irreducible,  $A$  is uniquely determined up to the scalar multiple and so we have  $\dim I_{M_D} \begin{pmatrix} U \\ M_{\alpha+D} & X \end{pmatrix} \leq 1$ . If  $\dim I_{M_D} \begin{pmatrix} U \\ M_{\alpha+D} & X \end{pmatrix} = 1$ , then as we showed the  $\hat{K}$ -module  $Q^\beta$  is isomorphic to  $Q^\gamma$  and so the  $M_D$ -module structure on  $U$  is uniquely determined by Theorem 5.3. ■

## 6 Hamming code VOA $V_{H_8}$

### 6.1 Definition of $V_{H_8}$ and its twenty four conformal vectors

An interesting property of the above construction is that the full automorphism group of  $M_D$  has a normal subgroup which is a 3-transposition group. A 3-transposition group is a group generated by a conjugacy class of involutions and the product of any two involutions in this class has the order less than or equal to three. Especially, if  $D$  contains a  $[8, 4, 4]$ -Hamming code  $H_8$  as a subspace, it defines new sixteen rational conformal vectors of central charge  $\frac{1}{2}$  and so we have new automorphisms given by conformal vectors (see



[M2]).

**Definition 11** *An even binary linear code  $\mathcal{H}_8$  with the generator matrix*

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \quad (6.1)$$

*is called the  $[8, 4, 4]$  Hamming code.*

We can construct a VOA  $M_{H_8}$  from  $[8, 4, 4]$ -Hamming code  $H_8$  by the tensor products of SVOA  $M$  and denote it by  $V_{H_8}$ . Let  $\{e^1, \dots, e^8\}$  be a set of coordinate conformal vectors.

In  $\mathcal{H}_8$ , there are fourteen codewords of weight four. By the direct calculation, we can obtain the following:

**Lemma 6.1** *For a four-point set  $\xi$ , we write  $u^\xi = q^\xi \otimes e^\xi$  in this section to simply the notation. Let  $\alpha$  and  $\beta$  be four-point sets. Then  $u^\alpha, u^\beta \in (V_{H_8})_2$  and we have:*

$$\begin{aligned} (u^\alpha)_1(u^\alpha) &= 2(\sum_{i \in \alpha} e^i) \\ (u^\alpha)_1 u^\beta &= u^{\alpha+\beta} && \text{if } |\alpha \cap \beta| = 2 \\ (u^\alpha)_1 u^\beta &= 0 && \text{if } \alpha \cap \beta = \emptyset \\ \langle e^i, e^j \rangle &= \frac{1}{4} \delta_{i,j} && \text{and} \\ \langle u^\alpha, u^\beta \rangle &= \delta_{\alpha, \beta}. \end{aligned} \quad (6.2)$$

As we showed in [M2], we can construct conformal vectors as follows:

**Theorem 6.1** *In  $V_{H_8}$ , we have the following conformal vectors with central charge  $\frac{1}{2}$ :*

$$s^\alpha = \frac{1}{8}(e^1 + \dots + e^8) + \frac{1}{8} \sum_{\beta \in C, |\beta|=4} (-1)^{\langle \alpha, \beta \rangle} q^\beta \otimes e^\beta \quad (6.3)$$

for a word  $\alpha$ . This is defined by the co-code  $\mathbf{Z}_2^8/H_8$ , that is,  $s^\alpha = s^\beta$  if and only if  $\alpha - \beta \in H_8$ . In particular, we have sixteen new rational conformal vectors. By the direct calculation using Lemma 6.1, we have  $\langle s^\alpha, s^\beta \rangle = 0$  if and only if  $\alpha + \beta$  has even weight.

Namely, we obtain the following three sets of eight mutually orthogonal conformal vectors of  $V_{H_8}$ .

$$\{e^1, \dots, e^8\}, \quad \{s^\beta : |\beta| \text{ odd weight}\}, \quad \{s^\alpha : |\alpha| \text{ even weight}\}. \quad (6.4)$$

We can take  $\{\nu_1 = (10 \cdots 0), \nu_2 = (010 \cdots 0), \dots, \nu_8 = (0 \cdots 01)\}$  as the set of representatives of the odd weight cosets of  $\mathbf{Z}_2^8/H_8$  and  $\{\nu_1 + \nu_1 = (0 \cdots 0), \nu_1 + \nu_2, \dots, \nu_1 + \nu_8 = (10 \cdots 01)\}$  as that of even weight cosets of  $\mathbf{Z}_2^8/H_8$ .

**Definition 12** *We will use the following notation in this paper.*

$$\begin{aligned} d^j &= s^{\nu_1 + \nu_j} & \text{for } j = 1, \dots, 8 \text{ and} \\ f^i &= s^{\nu_i} & \text{for } i = 1, \dots, 8. \end{aligned} \quad (6.5)$$

By the direct calculation using Lemma 6.1 and the results in [M1], we obtain the following lemma.

**Lemma 6.2** *There are exactly three sets of mutually orthogonal 8 conformal vectors with central charge  $\frac{1}{2}$  in  $V_{H_8}$ .*

**[Proof]** Clearly, since  $H_8$  has no codeword of weight 2,  $(V_{H_8})_1 = 0$  by the definition of code VOA. Suppose false and let  $\{g^1, \dots, g^8\}$  be another set of 8 conformal vectors. Viewing  $V_{H_8}$  as a  $\langle g^1, \dots, g^8 \rangle$ -module, there is no  $L(\frac{1}{2}, \frac{1}{16})$  since  $\sum g^i$  is the Virasoro element and the weights of elements in  $V_{H_8}$  are integers. Therefore,  $g^i$  are all of type 2 and so we have  $\langle e^i, g^j \rangle = 1/32$ . Since

$$(V_{H_8})_2 = \mathbf{C}e^1 + \cdots + \mathbf{C}e^8 + \sum_{\beta \in C, |\beta|=4} \mathbf{C}u^\beta,$$

if we set  $g = \sum a^i e^i + \sum a^\beta u^\beta$ , then we have  $a_i = 1/8$ . Calculating  $\langle g^i, g^i \rangle$  and  $g_1^i g^i$  by using (6.2), we have  $\{g^1, \dots, g^8\}$  is equal to  $\{f^1, \dots, f^8\}$  or  $\{d^1, \dots, d^8\}$ . ■

It is a routine work to see the action of  $\sigma_{e_i}$  on  $s^\alpha$ .

**Lemma 6.3**  $\sigma_{e_i}(s^\alpha) = s^{\alpha + \nu_i}$ .

[**Proof**]

$$\begin{aligned}
\sigma_{e^i}(s^\alpha) &= \sigma_{e^i} \left\{ \frac{1}{8}(\sum e^i) + \frac{1}{8}(\sum (-1)^{<\alpha, \beta>} u^\beta) \right\} \\
&= \frac{1}{8} \sum e^i + \sum (-1)^{<\alpha, \beta>} (-1)^{<\nu_i, \beta>} u^\beta \\
&= \frac{1}{8} \sum e^i + \sum (-1)^{<\alpha + \nu_i, \beta>} u^\beta \\
&= s^{\nu_i + \alpha}.
\end{aligned}$$

Namely, each involution  $\sigma_{e^i}$  permutes the new sixteen conformal vectors regularly. We want to note one more thing. Since  $[8, 4, 4]$ -Hamming code is the only one self-dual doubly even code of length 8,  $V_{H_8}$  is still isomorphic to  $V_{H_8}$  as  $\langle f^1, \dots, f^8 \rangle$ -modules.

## 6.2 Irreducible $V_{H_8}$ -modules with $\mathbf{Z}/2$ lowest weights

Let  $V_{H_8}$  be a code VOA constructed from  $H_8$  and  $\{e^1, \dots, e^8\}$  the set of coordinate conformal vectors. In this subsection, we will study the fusion rules of irreducible  $V_{H_8}$ -modules having lowest weights in  $\frac{1}{2}\mathbf{Z}$ . Let  $W$  be an irreducible  $V_{H_8}$ -module and  $\beta$  its  $\frac{1}{16}$ -word. By Lemma 5.1,  $\beta \in H_8$ . In particular,  $|\beta| \equiv 0 \pmod{4}$  and so the lowest weight of  $W$  is in  $\frac{1}{4}\mathbf{Z}$ .

If  $L$  is an irreducible  $V_{H_8}$ -module with integer lowest weight, then there is no  $\frac{1}{16}$  entry in a lowest weight row since the sum of entries in a lowest weight row is an integer and so we conclude  $L = \sum_\alpha M_\alpha$  as  $T$ -modules. Since the  $\frac{1}{16}$ -word of  $L$  is  $(0^8)$ , a  $T$ -module structure on  $L$  determines the  $V_{H_8}$ -module structure uniquely and so  $L = M_{\alpha+H_8}$  for some  $\alpha \in \mathbf{Z}_2^8$ .

Before we treat irreducible modules whose lowest weights are half-integers, we will prove the following lemma by the similar argument as in Theorem 6.12 in [M1].

**Lemma 6.4** *Let  $V = \sum_{i=0}^\infty V_i$  be a VOA over  $\mathbf{R}$  satisfying  $\dim V_0 = 1$  and  $V_1 = 0$ . Assume that  $V$  has a positive definite invariant bilinear form  $(*, *)$  and  $e$  and  $f$  are two distinct non-orthogonal rational conformal vectors with central charge  $\frac{1}{2}$ . Assume further that  $\tau_e$  fixes  $f$  and  $\tau_f$  fixes  $e$ . If  $L$  is a  $V$ -module such that  $L$  is isomorphic to a direct sum of  $L(\frac{1}{2}, \frac{1}{16})$  as  $\langle e \rangle$ -modules and  $\langle f \rangle$ -modules, then there is a conformal vector  $d$  with central charge  $\frac{1}{2}$  such that  $L$  has no  $L(\frac{1}{2}, \frac{1}{16})$  as  $\langle d \rangle$ -modules.*

[**Proof**] As in the proof of lemma 6.11 in [M1], we have  $\alpha, \beta \in V_2$  satisfying  $e\alpha = 0$  and  $e\beta = \frac{1}{2}\beta$  such that  $f = \lambda e + \alpha + \beta$ . In the proof of Lemma 6.11 in [M1], we proved that  $\lambda = \frac{1}{8}$  and  $\frac{16}{5}\alpha$  is a conformal vector with central charge  $\frac{7}{10}$ . Since  $\alpha_1$  and  $e_1$  acts on  $L$ , so does  $\beta_1$ . It is easy to see that  $d = \frac{1}{8}e + \alpha - \beta$  is also a conformal vectors with central charge

$\frac{1}{2}$ . Since the lowest weights of  $\frac{16}{5}\alpha$  is one of  $0, \frac{3}{80}, \frac{8}{80}, \frac{35}{80}, \frac{48}{80}, \frac{120}{80}$  by the representation of  $L(\frac{7}{10}, 0)$ ,  $d$  cannot have  $\frac{1}{16}$  as an eigenvalue on  $L$ . Hence,  $d$  is a conformal vector satisfying the desired conditions.  $\blacksquare$

If  $L$  is an irreducible  $V_{H_8}$ -module with half-integer lowest weight, then  $L = (\sum_{\alpha} M_{\alpha}) + (\sum \otimes_{i=1}^8 L^i(\frac{1}{2}, \frac{1}{16}))$  as  $T$ -modules and so we have  $L \cong M_{\alpha+H_8}$  or  $L \cong \sum \otimes_{i=1}^8 L^i(\frac{1}{2}, \frac{1}{16})$  as  $T$ -modules by the fusion rules (1.1). If  $L \cong M_{\alpha+H_8}$  as  $T$ -modules, then  $L \cong M_{\alpha+H_8}$  as  $V_{H_8}$ -modules as in the case of integer lowest weight. If  $L = \sum \otimes_{i=1}^8 L^i(\frac{1}{2}, \frac{1}{16})$ , then there is a set of mutually orthogonal conformal vectors, say  $\{d^1, \dots, d^8\}$ , such that  $L$  is isomorphic to  $M_{\alpha+H_8}$  for some  $\alpha \in \mathbf{Z}_2^8$  as  $\langle d^1, \dots, d^8 \rangle$ -modules by Lemma 6.4. So, we have the following classification.

**Theorem 6.2** *If  $L$  is an irreducible  $V_{H_8}$ -module with half-integer or integer lowest weight, then  $L$  is isomorphic to one of the followings:*

- (1)  $M_{\nu_1+\nu_i+H_8}$  for  $i = 1, \dots, 8$  as  $\langle e^1, \dots, e^8 \rangle$ -modules,
- (2)  $M_{\nu_i+H_8}$  for  $i = 1, \dots, 8$  as  $\langle e^1, \dots, e^8 \rangle$ -modules,
- (3)  $M_{\nu_i+H_8}$  for  $i = 1, \dots, 8$  as  $\langle f^1, \dots, f^8 \rangle$ -modules, and
- (4)  $M_{\nu_i+H_8}$  for  $i = 1, \dots, 8$  as  $\langle d^1, \dots, d^8 \rangle$ -modules.

Here  $\nu_i$  denotes the word  $(0, \dots, 0, 1, 0, \dots, 0)$  whose  $i$ th entry is 1 and the others are all 0. All irreducible modules in (3) and (4) are isomorphic to  $\otimes_{i=1}^8 L(\frac{1}{2}, \frac{1}{16})$  as  $\langle e^1, \dots, e^8 \rangle$ -modules.

[**Proof**] We have already proved that  $L$  is isomorphic to one of the modules in the list. We also showed that  $M_{\nu_i+H_8}$  is isomorphic to  $\otimes_{i=1}^8 L(\frac{1}{2}, \frac{1}{16})$  as  $\langle f^1, \dots, f^8 \rangle$ -modules in [M2].  $\blacksquare$

**Definition 13** *We will use  $H(h^i, \alpha)$  to denote the above modules, that is,*

$$\begin{aligned} H(\frac{1}{2}, \alpha) &\cong M_{\alpha+H_8} && \text{as } \langle e^1, \dots, e^8 \rangle\text{-modules as in (1) and (2),} \\ H(\frac{1}{16}, \alpha) &\cong M_{\nu_i+H_8} && \text{as } \langle f^1, \dots, f^8 \rangle\text{-modules as in (3) if } f^i = s^{\alpha} \text{ and} \\ H(\frac{1}{16}, \beta) &\cong M_{\nu_i+H_8} && \text{as } \langle d^1, \dots, d^8 \rangle\text{-modules as in (4) if } d^i = s^{\beta}, \end{aligned}$$

where  $s^{\alpha}$  is given in (6.3). Consequently, we have exactly thirty two irreducible  $V_{H_8}$ -modules with lowest weights in  $\mathbf{Z}/2$ .

The next lemma is clear by the definition.

**Lemma 6.5**  $H(h, \alpha) \cong H(k, \beta)$  if and only if  $h = k$  and  $\alpha - \beta \in H_8$ .

**Remark 7** The lowest weights of modules in (2), (3), (4) are  $\frac{1}{2}$  and those of the modules in (1) are one. The modules  $H(\frac{1}{16}, \alpha)$  is characterized by the following properties:

(i)  $H(\frac{1}{16}, \alpha) \cong \otimes L(\frac{1}{2}, \frac{1}{16})$  as  $T$ -modules.

(ii) The vertex operator of  $u^\beta$  on  $H(\frac{1}{16}, \alpha)$  is  $(-1)^{\langle \alpha, \beta \rangle} I(u^\beta, z)$ .

### 6.3 Fusion rule of $V_{H_8}$

In this section, we will determine the fusion rules of the irreducible  $V_{H_8}$ -modules with lowest weights in  $\mathbf{Z}/2$ . In the previous section, we showed that such irreducible modules are

$$\left\{ H(\frac{1}{2}, \alpha), H(\frac{1}{16}, \beta) : \alpha, \beta \in \mathbf{Z}_2^8/H_8 \right\}. \quad (6.6)$$

We first prove the next lemma.

**Lemma 6.6** If  $\alpha$  is an even word, then  $H(\frac{1}{2}, \alpha) \cong M_{\alpha+H_8}$  as  $\langle f^1, \dots, f^8 \rangle$ -modules. It is also true for  $\langle d^1, \dots, d^8 \rangle$ .

**[Proof]** By Lemma 6.3 and Definition 12, we have  $\sigma_{e_1}(f^i) = d^i$  for all  $i$  and  $\sigma_{e_1}$  fixes all  $H(\frac{1}{2}, \alpha)$ . Hence, it is sufficient to prove the lemma for  $\langle f^1, \dots, f^8 \rangle$ . Without loss of generality, we may assume  $\alpha = \{1, 2\}$ . The lowest weight of  $H(\frac{1}{2}, \alpha)$  is one and the lowest weight space has a basis  $\{u^{\{12\}}, u^{\{34\}}, u^{\{56\}}, u^{\{78\}}\}$ , where  $\{1234\}, \{1256\}, \{1278\}$  are the set of four point codewords of  $H_8$  containing  $\{12\}$ . Hence,  $H(\frac{1}{2}, \alpha) \cong M_{\beta+H_8}$  as  $\langle f^1, \dots, f^8 \rangle$ -modules for some  $\beta$ . By the direct calculation of the eigenvalues of  $s^\beta = \frac{1}{8}\mathbf{w} + \frac{1}{8}\sum_{\gamma \in H_8, |\gamma|=4} (-1)^{\langle \beta, \gamma \rangle} u^\gamma$  on  $H(\frac{1}{2}, \alpha)$ , the eigenvalues of  $s^{\nu_1}$  and  $s^{\nu_2}$  on  $-u^{\{12\}} + u^{\{34\}} + u^{\{56\}} + u^{\{78\}}$  are the same  $\frac{1}{2}$  and those of others are 0. Hence,  $H(\frac{1}{2}, \{12\}) = M_{\{12\}+H_8}$  as  $\langle f^1, \dots, f^8 \rangle$ -modules. ■

By Theorem 5.5 and Theorem 6.2, we have the following theorem.

**Theorem 6.3** If  $W$  is an irreducible  $V_{H_8}$ -module, then the tensor product  $H(h, \beta) \times W$  is an irreducible module for any  $h$  and  $\beta$ .

**Remark 8** Let  $v \in H(h, \beta)$ . By the direct calculation, we have

$$\begin{aligned} & (z_1 - z_2)^N Y(s^\alpha, z_1) Y(u^\gamma, z_2) v \\ &= \frac{1}{8} (z_1 - z_2)^N \left\{ Y(\hat{\mathbf{w}}, z_1) Y(u^\gamma, z_2) + \sum_{\beta \in C, |\beta|=4} (-1)^{\langle \alpha, \beta \rangle} Y(u^\beta, z_1) Y(u^\gamma, z_2) \right\} v \\ &= \frac{1}{8} (z_1 - z_2)^N \left\{ Y(u^\gamma, z_2) Y(\hat{\mathbf{w}}, z_1) + \sum_{\beta \in C, |\beta|=4} (-1)^{\langle \alpha+\gamma, \beta \rangle} Y(u^\gamma, z_2) Y(u^\beta, z_1) \right\} v \\ &= (z_1 - z_2)^N Y(u^\gamma, z_2) Y(s^{\alpha+\gamma}, z_1) v \end{aligned} \quad (6.7)$$

Hence, the action of  $Y(s^\alpha, z)$  on  $Y(u^\gamma, z)H(h, \beta)$  is same as  $Y(s^{\alpha+\gamma}, z)$  on  $H(h, \beta)$ . So we have all actions of  $s^\alpha$  on  $H(\frac{1}{2}, \gamma) \times U$ .

As a corollary of Theorem 6.3 and Remark 8, we have:

**Corollary 6.1** *The fusion rules of  $V_{H8}$ -modules  $\{H(\frac{1}{2}, \alpha), H(\frac{1}{16}, \alpha)\}$  are*

$$\begin{aligned} H(\tfrac{1}{2}, \alpha) \times H(\tfrac{1}{2}, \beta) &= H(\tfrac{1}{2}, \alpha + \beta), \\ H(\tfrac{1}{2}, \alpha) \times H(\tfrac{1}{16}, \beta) &= H(\tfrac{1}{16}, \alpha + \beta) \quad \text{and} \\ H(\tfrac{1}{16}, \alpha) \times H(\tfrac{1}{16}, \beta) &= H(\tfrac{1}{2}, \alpha + \beta). \end{aligned} \tag{6.8}$$

**[Proof]** Changing the set of coordinate conformal vectors, it is sufficient to calculate the tensor product  $H(\frac{1}{2}, \alpha) \times H(h, \beta)$ . We have already proved that  $H(\frac{1}{2}, \alpha) \times H(h, \beta)$  is an irreducible module by Theorem 5.5. Since the action of  $Y(s^\gamma, z)$  on  $Y(u^\alpha, z)H(h, \beta)$  is same as  $Y(s^{\gamma+\alpha}, z)$  on  $H(h, \beta)$  by Remark 8, we have the desired fusion rules. ■

**Theorem 6.4** *If  $h = \frac{1}{2}$  and  $\alpha$  is an even set and  $\alpha \notin H_8$ , then  $V_{H8} \oplus H(h, \alpha)$  has a simple VOA structure on it. If  $h = \frac{1}{16}$  or  $\alpha$  is odd, then  $V_{H8} \oplus H(h, \alpha)$  has a simple SVOA structure on it.*

**[Proof]** Since the fusion rule  $H(h, \alpha) \times H(h, \alpha) = V_{H8}$  has a single component, the structure is unique. If  $h = \frac{1}{2}$  and  $\alpha$  is even set, then  $V = V_{H8} \oplus H(\frac{1}{2}, \alpha)$  is a subspace of  $M_S$ , where  $S$  is the code consisting of all even words. Since  $S$  is an even linear code,  $M_S$  has a VOA structure on it with positive definite invariant bilinear form, so does  $V$ . If  $h = \frac{1}{16}$  or  $\alpha$  is odd, then we may assume that  $h = \frac{1}{2}$  and  $\alpha$  is odd by Definition 13. For an odd word  $\alpha$ ,  $V_{H8} \oplus H(\frac{1}{2}, \alpha)$  is a subspace of  $M_{\mathbf{Z}_2^n}$  and  $M_S \cap (V_{H8} \oplus H(\frac{1}{2}, \alpha)) = V_{H8}$ . Since  $M_{\mathbf{Z}_2^n}$  has a SVOA structure on it with the even part  $M_S$  and the odd part  $M_{S+\alpha}$ , we have the desired conclusions. ■

## 6.4 VOA structure and fusion rules

In this subsection,  $E$  denotes a self-orthogonal subcode of  $D$  and we assume that  $E$  is a direct sum  $E = \oplus E^i$  of Hamming codes  $E^i$  and  $U = \otimes H(h^i, \alpha^i)$  is an  $M_E$ -module such that  $M_E \oplus U$  has a simple VOA structure. We next show that the induced  $M_D$ -module also has a simple VOA structure.

**Theorem 6.5** *If  $M_E \oplus U$  has a simple VOA structure on it, then so does  $M_D \oplus \text{Ind}_{M_E}^{M_D}(U)$ .*

**[Proof]** Let  $(M_D, Y^0)$  be a code VOA and  $(M_E \oplus U, Y^1)$  a simple VOA. Since  $U \times U = M_E$  and  $U \times M_E = U$ , the vertex operator  $Y^1(u, z)$  of  $u \in U$  has an expression

$$Y^1(u, z) = \begin{pmatrix} 0 & I^1(u, z) \\ I^2(u, z) & 0 \end{pmatrix}, \quad (6.9)$$

where  $I^1(u, z) \in \text{Hom}(U, M_E)\{z, z^{-1}\}$ , and  $I^2(u, z) \in \text{Hom}(M_E, U)\{z, z^{-1}\}$ .

Set  $W = \text{Ind}_{M_E}^{M_D}(U)$  and let  $Y^W$  be an induced module vertex operator. We first show  $W \times W = M_D$ . Let  $U^0$  be an  $M_D$ -module with  $I_{M_D} \begin{pmatrix} U^0 \\ W & W \end{pmatrix} \neq 0$ . Then  $I_{M_E} \begin{pmatrix} U^0 \\ U & U \end{pmatrix} \neq 0$  by Theorem 5.4 and so  $U^0$  contains  $M_E = U \times U$  as  $M_E$ -modules. Since the  $M_D$ -module structure of  $U^0$  is determined by an  $M_E$ -submodule by Theorem 5.3, we see  $U^0 = M_D$ . Since  $M_E$ -module  $M_D$  has only one irreducible submodule isomorphic to  $M_E$ , we have  $W \times W = M_D$  and there is a non-zero intertwining operator  $I(*, z) \in I \begin{pmatrix} M_D \\ W & W \end{pmatrix}$ . We will next define vertex operators

$$Y(v, z) \in \text{End}(M_D \oplus \text{Ind}_{M_E}^{M_D}(U))[[z, z^{-1}]]$$

of  $v \in W = \text{Ind}_{M_E}^{M_D}(U)$  satisfying the derivation and the mutually commutativity. We also have an intertwining operator  $I'(*, z) \in I \begin{pmatrix} W \\ W & M_D \end{pmatrix}$  since  $W \times M_D \cong M_D \times W$ . We therefore obtain a vertex operator

$$Y(v, z) = \begin{pmatrix} 0 & I(v, z) \\ I'(v, z) & 0 \end{pmatrix} \in \text{End}(M_D \oplus W)[[z, z^{-1}]] \quad (6.10)$$

of  $v \in W$ . By the properties of intertwining operators,  $Y(v, z)$  satisfies the derivation and the commutativity and the associativity with the vertex operators

$$Y(u, z) = \begin{pmatrix} Y^0(u, z) & 0 \\ 0 & Y^W(u, z) \end{pmatrix} \quad (6.11)$$

of  $u \in M_D$ . By the construction,  $M_E$ -module  $\text{Ind}_{M_E}^{M_D}(U)$  contains only one irreducible submodule isomorphic to  $U$  and  $U \times U = M_E$ . Hence, if we restrict this vertex operator to that of  $v \in M_E \oplus U$ , then it should be equal to the vertex operator  $Y^1(v, z) \in \text{End}(M_E \oplus U)[[z, z^{-1}]]$  of  $v \in M_E \oplus U$  by multiplying some scalars to  $I(v, z)$  and  $I'(v, z)$ . In particular, there is a sufficiently large integer  $N$  such that

$$(z_1 - z_2)^N Y(v, z_1) Y(v, z_2) w = (z_1 - z_2)^N Y(v, z_2) Y(v, z_1) w \quad (6.12)$$

for  $w \in M_E \oplus U$ . Since  $Y(v, z)$  satisfies the commutativity with  $Y(u, z)$  of elements  $u \in M_D$ , we have

$$\begin{aligned}
& (z_1 - z_2)^N (z_1 - z_3)^N (z_2 - z_3)^N Y(v, z_1) Y(v, z_2) Y(u, z_3) w \\
&= (z_1 - z_2)^N (z_1 - z_3)^N (z_2 - z_3)^N Y(u, z_3) Y(v, z_1) Y(v, z_2) w \\
&= (z_1 - z_2)^N (z_1 - z_3)^N (z_2 - z_3)^N Y(u, z_3) Y(v, z_2) Y(v, z_1) w \\
&= (z_1 - z_2)^N (z_1 - z_3)^N (z_2 - z_3)^N Y(v, z_2) Y(v, z_1) Y(u, z_3) w
\end{aligned} \tag{6.13}$$

for a sufficiently large integer  $N$ . Hence, by multiplying a power of  $z_3$  and substituting 0 into  $z_3$ , we obtain

$$(z_1 - z_2)^N Y(v, z_1) Y(v, z_2) u_r w = (z_1 - z_2)^N Y(v, z_2) Y(v, z_1) u_r w, \tag{6.14}$$

where  $u_r w \neq 0$  and  $u_{r+i} w = 0$  for all  $i > 0$ . Substituting this into the above, we see

$$(z_1 - z_2)^N Y(v, z_1) Y(v, z_2) u_i w = (z_1 - z_2)^N Y(v, z_2) Y(v, z_1) u_i w \tag{6.15}$$

for all  $i$ . Since  $\{u_i w : u \in M_D, w \in M_E \oplus U\}$  spans whole space  $M_D \oplus W$  by Proposition 4.1 in [DM],  $Y(v, z)$  satisfies the commutativity with itself. It is clear that it satisfies the other conditions required to be a vertex operator of VOA. We note that we don't need to prove the associativity of  $Y(*, z)$ . It is also clear that  $(V, Y)$  is simple. ■

Let  $X$  be an irreducible  $M_D$ -module and assume that an irreducible  $M_E$ -submodule of  $X$  is isomorphic to  $\otimes_{i=1}^k H(h^i, \beta^i)$  for some  $h^i \in \{\frac{1}{2}, \frac{1}{16}\}$  and  $\beta^i \in \mathbf{Z}_2^8$ , where  $n = 8k$ . We will calculate the fusion rules of  $X$  with others. Let  $\alpha^1$  be an  $\frac{1}{16}$ -word of  $X$ . In order to simply the notation, we assume  $\alpha^1 = (1^{8s} 0^{8(k-s)})$ . Namely, we will assume the following:

### Hypotheses I

- (1) Let  $D$  be an even binary linear code with length  $8k$  and  $E$  a self-orthogonal subcode of  $D$ .
- (2) Assume that  $E$  is a direct sum  $\oplus_{i=1}^k E^i$  of Hamming codes  $E^i$ .
- (3) Let  $U$  be an  $M_E$ -module  $(H(\frac{1}{16}, \alpha^1) \otimes \cdots \otimes H(\frac{1}{16}, \alpha^s)) \otimes (H(\frac{1}{2}, \alpha^{s+1}) \otimes \cdots \otimes H(\frac{1}{2}, \alpha^k))$ . and set  $M^1 = \text{Ind}_{M_E}^{M_D}(U)$ . We note that  $\alpha^1 = (1^{8s} 0^{8(k-s)})$  is a  $\frac{1}{16}$ -word of  $M^1$ .
- (4) Let  $M^2$  be an irreducible  $M_D$ -module with a  $\frac{1}{16}$ -word  $\alpha^2$ . Set  $\alpha^3 = \alpha^1 + \alpha^2$ .
- (5) Set  $K^i = \{\beta \in D | \beta \subseteq \alpha^i\}$  and assume  $E + K^2 = E + K^3$ . We note that  $K^1$  contains  $\oplus_{i=1}^s E^i$ .

**Remark 9** We will explain the meaning of the last assumption. Choose  $\alpha \in K^2$ , then there is an  $\beta_\alpha \in E$  such that  $\alpha + \beta_\alpha \in K^3$ . Set  $\gamma = \alpha^1 \cap \alpha^2$ ,  $\gamma^1 = \alpha^1 - \gamma$  and  $\gamma^2 = \alpha^2 - \gamma$ .



The above relation implies that  $\alpha \cap \tilde{h} = \beta_\alpha \cap \tilde{h}$  since  $(\alpha + \beta_\alpha) \cap \gamma = \emptyset$ . Since  $\alpha + \beta_\alpha \in K^3$  and  $\alpha^1 + \lambda^2 \cap \gamma = \emptyset$ , we have  $\alpha + \beta_\alpha \subseteq \alpha^3 = \alpha^1 + \alpha^2$  and so  $\alpha + \beta_\alpha \cap \gamma = \emptyset$ . If  $\alpha \in K^2$ , then  $\alpha = (\alpha \cap \gamma) \cup (\alpha \cap \gamma^2)$  and  $\gamma \cap \gamma^1 = \emptyset$ . Hence, for  $\alpha, \xi \in K^2$ , we obtain

$$\langle \alpha, \gamma \rangle = \langle \alpha \cap \gamma, \xi \cap \gamma \rangle + \langle \alpha \cap \gamma^2, \xi \cap \gamma^2 \rangle.$$

Similarly, we have

$$0 = \langle \beta_\alpha, \beta_\xi \rangle = \langle \beta_\alpha \cap \gamma^1, \beta_\xi \cap \gamma^1 \rangle + \langle \beta_\alpha \cap \gamma, \beta_\gamma \cap \gamma \rangle$$

for  $\beta_\alpha, \beta_\xi \in E$  since  $E$  is self-orthogonal. Hence, if  $\langle \alpha, \gamma \rangle = 0$  for  $\alpha, \xi \in K^2$ , then

$$\langle \alpha \cap \gamma, \xi \cap \gamma \rangle = \langle \alpha \cap \gamma^2, \xi \cap \gamma^2 \rangle$$

and

$$\langle \beta_\alpha \cap \gamma^1, \beta_\xi \cap \gamma^1 \rangle = \langle \beta_\alpha \cap \gamma, \beta_\gamma \cap \gamma \rangle.$$

Therefore, we obtain

$$\begin{aligned} \langle \alpha + \beta_\alpha, \xi + \beta_\xi \rangle &= \langle \alpha \cap \gamma^2 + \beta_\alpha \cap \gamma^1, \xi \cap \gamma^2 + \beta_\xi \cap \gamma^1 \rangle \\ &= \langle \alpha \cap \gamma^2, \xi \cap \gamma^2 \rangle + \langle \beta_\alpha \cap \gamma^1, \beta_\xi \cap \gamma^1 \rangle = 0. \end{aligned}$$

Namely, if  $H^2$  is a self-orthogonal subcode of  $K^2$ , then  $\{\beta_\alpha : \alpha \in H^2\}$  is also a self-orthogonal. In particular, if  $H^2$  is a maximal self-orthogonal subcode of  $K^2$ , then there is a maximal self-orthogonal subcode  $H^3$  of  $K^3$  such that  $E + H^2 = E + H^3$ .

We will prove the one of our main theorems.

**Theorem 6.6** *Under Hypotheses I,  $M^1 \times M^2$  is irreducible.*

**[Proof]** Let  $M^3$  be an irreducible  $M_D$ -module such that  $I_{M_D} \begin{pmatrix} M^3 \\ M^1 & M^2 \end{pmatrix} \neq 0$  and  $U^2$  an irreducible  $M_E$ -submodule of  $M^2$ . Clearly,  $\alpha^3$  is the  $\frac{1}{16}$ -word of  $M^3$ . Let  $H^i$  be maximal self-orthogonal subcode of  $K^i$  such that  $E + H^2 = E + H^3$  by Remark 9. Assume first that  $D = E + H^3$ . Then  $M^2$  and  $M^3$  are both irreducible  $M_E$ -modules by Theorem 5.1 and we have

$$\dim I_{M_D} \begin{pmatrix} M^3 \\ M^1 & M^2 \end{pmatrix} \leq \dim I_{M_E} \begin{pmatrix} M^3 \\ U & M^2 \end{pmatrix} = 1$$

by Theorem 5.4. We therefore obtain  $M^3 = U \times M^2$  as  $M_E$ -modules. Choose a nonzero intertwining operator  $I^1(*, z) \in I_{M_E} \begin{pmatrix} M^3 \\ U & M^2 \end{pmatrix}$ . Then for  $I(v, z) \in I_{M_D} \begin{pmatrix} M^3 \\ M^1 & M^2 \end{pmatrix}$ ,

there is a scalar  $\lambda$  such that  $I(v, z) = \lambda I^1(v, z)$  for  $v \in U$ . By the commutativity of intertwining operator

$$0 = (z_1 - z_2)^N Y^3(u, z_1) I(v, z_2) w = (z_1 - z_2)^N I(v, z_2) Y^2(u, z_1) w,$$

we have

$$0 = (z_1 - z_2)^N \lambda Y^3(u, z_1) I^1(v, z_2) w = (z_1 - z_2)^N \lambda I^1(v, z_2) Y^2(u, z_1) w$$

for all  $v \in U$ ,  $u \in M_D$  and  $w \in M^2$  and for a sufficiently large integer  $N$ . Hence, the action  $Y^3(u, z)$  of  $u \in M_D$  on  $M^3$  does not depend on the choice of  $\lambda$  and it is determined by  $Y^2(u, z)$  and  $I^1(v, z_2)$ . Therefore, the structure of  $M^3$  is uniquely determined and  $M^1 \times M^2$  is irreducible.

We will next prove the general case. Assume  $H = E + H^3 \neq D$ . Let  $U^1$  and  $U^2$  be irreducible  $M_H$ -submodules of  $M^1$  and  $M^2$ , respectively. Then the tensor product  $U^1 \times U^2$  is an irreducible  $M_H$ -module as we showed and hence  $M^3$  contains  $U^1 \times U^2$  as  $M_H$ -submodule. Therefore the  $M_D$ -module structure on  $M^3$  is uniquely determined by Theorem 5.3 and so  $M^1 \times M^2$  is irreducible. ■

## 7 Construction and uniqueness of VOA structure

In this section, we will show a new construction of vertex operator algebras. We will define a Fock space  $V$  containing  $M_D$  and then show that the vertex operators of elements will be automatically determined by its representations.

### 7.1 The setting

We will assume the following Hypotheses II (1)  $\sim$  (8).

#### Hypotheses II

- (1)  $D$  and  $S$  are both even linear codes with length  $8k$ .
- (2)  $\langle D, S \rangle = 0$ .
- (3) For all  $\alpha \in S$ , the weight  $|\alpha|$  is a multiple of eight.
- (4) For any  $\alpha \in S$ ,  $D$  contains a self-dual subcode  $E_\alpha$  such that  $E_\alpha$  is a direct sum  $E_\alpha = \bigoplus_{i=1}^k E_\alpha^i$  of Hamming codes  $E_\alpha^i$ . Assume that  $H_\alpha = \{\beta \in E_\alpha : \beta \subseteq \alpha\}$  is a direct factor of  $E_\alpha$  containing  $\alpha$ .
- (5) Set  $K_\alpha = \{\beta \in D : \beta \subseteq \alpha\}$  and assume that for any two  $\alpha, \beta \in S$ ,  $K_\alpha + H_\beta =$

$$K_{\alpha+\beta} + H_\beta.$$

Let  $\{\alpha^1, \dots, \alpha^t\}$  be a basis of  $S$ . For each  $j$ ,  $E_{\alpha^j}$  is a direct sum  $\oplus_{i=1}^k E_{\alpha^j}^i$  of Hamming codes and  $U_{\alpha^j} = \otimes_{i=1}^k H(h_i^j, \beta_i^j)$  is an  $M_{E_{\alpha^j}}$ -module with a  $\frac{1}{16}$ -word  $\alpha^j$ .

(6) We assume that  $M_{E_{\alpha^j}} \oplus U_{\alpha^j}$  has a simple VOA structure on it.

As we showed in Theorem 6.4, if  $\beta$  is even, then  $V_{H8} \oplus H(\frac{1}{2}, \beta)$  has a VOA structure and if  $h = \frac{1}{16}$  or  $\beta$  is odd, then  $V_{H8} \oplus H(h, \beta)$  has a SOVA structure. Hence, (6) is equivalent to :

$$(6') \quad |\{i : h_i^j = \frac{1}{16}, \alpha_i^j \text{ is odd}\}| \equiv 0 \pmod{2} \text{ for all } j.$$

Set

$$V^{\alpha^i} = \text{Ind}_{M_{E^i}}^{M_D}(U_{\alpha^i}).$$

This is an irreducible  $M_D$ -module. By Theorem 6.5,  $M_D \oplus V^{\alpha^i}$  has a simple VOA structure.

By Theorem 6.6 and the above hypotheses,  $V^{\alpha^i} \times V^{\alpha^j}$  is irreducible. Set  $V^{\alpha^i + \alpha^j} = V^{\alpha^i} \times V^{\alpha^j}$ . By the symmetry of the fusion rule and the uniqueness, it is equal to  $V^{\alpha^j + \alpha^i}$ . In order to simplify the notation, we will denote  $\alpha^i + \alpha^j$  by  $\alpha$  for a while. The  $\frac{1}{16}$ -word of  $V^\alpha$  is  $\alpha$  and there is a self-orthogonal subcode  $E_\alpha$  in  $D$  such that  $E_\alpha$  is a direct sum of Hamming codes  $E_\alpha^i$  by the condition (4). Hence  $V^\alpha$  is also an induced module  $\text{Ind}_{M_{E_\alpha}}^{M_D}(U_\alpha)$  for an irreducible  $M_{E_\alpha}$ -module  $U_\alpha \cong \otimes H(h^j, \beta^j)$ . We can therefore define the tensor product  $V^{\alpha^k + \alpha^i + \alpha^j} = V^{\alpha^i + \alpha^j} \times V^{\alpha^k}$  uniquely. Repeating these steps, we can define all  $M_D$ -modules  $V^\alpha$  for  $\alpha \in S$  by

$$V^\alpha = (\dots (V^{\alpha^{j_1}} \times V^{\alpha^{j_2}}) \times \dots) \quad (7.1)$$

for  $\alpha = \alpha^{j_1} + \dots + \alpha^{j_r}$  with  $j_1 < \dots < j_r$ .

We assume the associativity:

(7)  $V^\alpha$  does not depend on the order of products.

It is easy to see that (7) is equivalent to the associativity of product of three elements:

$$(V^{\alpha^i} \times V^{\alpha^j}) \times V^{\alpha^k} = V^{\alpha^i} \times (V^{\alpha^j} \times V^{\alpha^k}). \quad (7.2)$$

By the definition of  $V^\alpha$  and the assumption (7), we have:

**Lemma 7.1**  $V^\alpha \times V^\beta = V^{\alpha+\beta}$ .

At last, we assume that the commutativity of intertwining operators:

(8) For  $I^\beta(*, z) \in I \begin{pmatrix} V^{\alpha+\beta} \\ V^\alpha & V^\beta \end{pmatrix}$  and  $I^{\beta+\alpha}(*, z) \in I \begin{pmatrix} V^\beta \\ V^\alpha & V^{\alpha+\beta} \end{pmatrix}$ , we assume that the powers of  $z$  in  $I^\beta(*, z)$  and  $I^{\beta+\alpha}(*, z)$  are all integers and they satisfy the commutativity, that is, for  $v \in V^\alpha$ , we assume

$$(z_1 - z_2)^N I^\beta(v, z_1) I^{\beta+\alpha}(v, z_2) = (z_1 - z_2)^N I^{\beta+\alpha}(v, z_2) I^\beta(v, z_1) \quad (7.3)$$

for sufficiently large integer  $N$ .

Set

$$V = \oplus_{\alpha \in S} V^\alpha, \quad (7.4)$$

where  $V^0 = M_D$ .

Under the above Hypotheses II (1)  $\sim$  (8), we will show that a vertex operator  $Y(v, z)$  for every element  $v \in V$  is automatically determined.

## 7.2 Construction of vertex operators

Let  $\dim S = t$  and set  $S_i = \langle \alpha^1, \dots, \alpha^i \rangle$  for  $i = 0, 1, \dots, t$ . Set  $V^i = \oplus_{\alpha \in S_i} V^\alpha$ . We will define a vertex operator  $Y(v, z) \in \text{End}(V)[[z, z^{-1}]]$  of  $v \in V^i$  inductively. Since  $V^\alpha$  are all  $M_D$ -modules, the vertex operators  $Y(v, z)$  of  $v \in V^0 = M_D$  on  $V$  are already determined and they satisfy the mutual commutativity. Assume the vertex operators  $Y(w, z) \in \text{End}(V)[[z, z^{-1}]]$  of elements  $w \in V^r$  are already determined and they satisfy the mutual commutativity and we will define  $Y(v, z) \in \text{End}(V)[[z, z^{-1}]]$  for  $v \in V^{r+1}$ . Decompose  $V^{r+1} = V^r \oplus W$  as  $V^r$ -modules, then  $W$  is an irreducible  $V^r$ -modules and  $W = \oplus_{\beta \in S_{r+1} - S_r} V^\beta$ . Set  $\alpha = \alpha^{r+1} \in S_{r+1} - S_r$  to simplify the notation. By the assumption (6) and Theorem 6.5,  $M_D \oplus V^\alpha$  has a simple VOA structure and denote it by

$$(M_D \oplus V^\alpha, Y^\alpha). \quad (7.5)$$

We first prove:

**Lemma 7.2**  $W \times W = V^r$  as  $V^r$ -modules.

[**Proof**] Let  $U$  be an irreducible  $V^r$ -submodule of  $W \times W$ . By the same arguments as in the proof of Theorem 5.4, we have

$$\dim I_{V^r} \begin{pmatrix} U \\ W & W \end{pmatrix} \leq \dim I_{M_D} \begin{pmatrix} U \\ V^\alpha & V^{\alpha+\beta} \end{pmatrix} \quad (7.6)$$

for any  $\beta \in S_r$  and hence  $U$  contains  $V^\alpha \times V^{\alpha+\beta} = V^\beta$  as  $M_D$ -modules. Therefore the  $V^r$ -structure on  $U$  is uniquely determined and so  $U \cong V^r$ .  $\blacksquare$

By the same arguments as in the above proof,  $\oplus_{\delta \in S_r} V^{\delta+\mu}$  is an irreducible  $V^r$ -module and we have:

$$W \times (\oplus_{\delta \in S_r} V^{\delta+\gamma}) = \oplus_{\delta \in S_r} V^{\delta+\gamma+\alpha}, \quad (7.7)$$

which is also an irreducible  $V^r$ -module. As we showed in the proof of Theorem 5.4, we can induce every intertwining operator in

$I_{V^r} \left( \begin{smallmatrix} \oplus_{\delta \in S_r} V^{\delta+\alpha+\gamma} \\ W & \oplus_{\delta \in S_r} V^{\delta+\gamma} \end{smallmatrix} \right)$  from one in  $I_{M_D} \left( \begin{smallmatrix} V^{\delta+\alpha} \\ V^\alpha & V^\delta \end{smallmatrix} \right)$ . Namely, we can choose a nontrivial intertwining operator

$I^\gamma(*, z) \in I_{V^r} \left( \begin{smallmatrix} \oplus_{\delta \in S_r} V^{\delta+\alpha+\gamma} \\ W & \oplus_{\delta \in S_r} V^{\delta+\gamma} \end{smallmatrix} \right)$  for each  $\gamma \in S/S_r$  such that

$I^\gamma(*, z)|_{V^\gamma} \in I_{M_D} \left( \begin{smallmatrix} V^{\alpha+\gamma} \\ V^\alpha & V^\gamma \end{smallmatrix} \right)$  and  $I^\gamma(*, z)|_{V^{\gamma+\alpha}} \in I_{M_D} \left( \begin{smallmatrix} V^\gamma \\ V^\alpha & V^{\alpha+\gamma} \end{smallmatrix} \right)$ .

Since  $I^\gamma(*, z)$  is an intertwining operator, it satisfies the commutativities:

$$(z_1 - z_2)^N Y^{\gamma+\alpha}(u, z_1) I^\gamma(v^\alpha, z_2) w = (z_1 - z_2)^N I^\gamma(v^\alpha, z_2) Y^\gamma(u, z_1) w \quad (7.8)$$

for  $v^\alpha \in V^\alpha$ ,  $u \in V^r$ ,  $w \in \oplus_{\delta \in S_r} V^{\gamma+\delta}$  and

$$(z_1 - z_2)^N Y^\gamma(u, z_1) I^{\gamma+\alpha}(v^\alpha, z_2) w' = (z_1 - z_2)^N I^{\gamma+\alpha}(v^\alpha, z_2) Y^{\gamma+\alpha}(u, z_1) w' \quad (7.9)$$

for  $v^\alpha \in V^\alpha$ ,  $u \in V^r$ ,  $w' \in \oplus_{\delta \in S_r} V^{\gamma+\alpha+\delta}$ . Since a VOA  $(M_D \oplus V^\alpha, Y^\alpha)$  is given,  $I^\alpha(*, z) \in I \left( \begin{smallmatrix} W \\ W & V^r \end{smallmatrix} \right)$  is uniquely determined by the property:

$$v_{-1}^\alpha \mathbf{1} = v^\alpha,$$

where  $I^\alpha(v^\alpha, z) = \sum v_n^\alpha z^{-n-1}$  for  $v^\alpha \in V^\alpha$ . Also, by the commutativity of  $Y^\alpha$  on  $M_D \oplus V^\alpha$ ,  $I^0(v^\alpha, z) \in I \left( \begin{smallmatrix} V^r \\ W & W \end{smallmatrix} \right)$  is uniquely determined.

Define a vertex operator  $Y(v^\alpha, z) \in \text{End}(V)[[z, z^{-1}]]$  of  $v^\alpha \in V^\alpha$  by

$$Y(v^\alpha, z) = \begin{pmatrix} 0 & I^0(v^\alpha, z) & & & \\ I^\alpha(v^\alpha, z) & 0 & & & \\ & & 0 & I^\gamma(v^\alpha, z) & \\ & & I^{\gamma+\alpha}(v^\alpha, z) & 0 & \\ & & & & \dots \\ & & & & & \dots \end{pmatrix}, \quad (7.10)$$

then it satisfies the commutativity with  $Y(w, z)$  for all  $w \in V^r$  as we showed. Moreover, the assumption (8) means the commutativity:

$$(z_1 - z_2)^N Y(v^\alpha, z_1) Y(v^\alpha, z_2) w = (z_1 - z_2)^N Y(v^\alpha, z_2) Y(v^\alpha, z_1) w \quad (7.11)$$

for any  $\alpha \in S, w \in V^\beta$ . Therefore,  $Y(v^\alpha, z)$  satisfies the commutativity with itself. Since every  $I^\gamma(v^\alpha, z)$  satisfies the derivation, so does  $Y(v^\alpha, z)$ . Using the normal product, we can define all vertex operators  $Y(v, z)$  of  $v \in V^{r+1}$ , which still satisfy the mutual commutativity and the derivation. We therefore have a VOA structure on  $V$ . We next show that the VOA structures on  $V$  is unique. Clearly, if  $V$  has a VOA structure containing  $M_D$  as a subVOA, then  $V$  has the above structure that we have constructed. Since we can modify the difference of scalar times of  $I^\gamma(v^\alpha, z)$  by multiplying the scalar to the bases of  $\oplus_{\delta \in S_r} V^{\delta+\gamma}$  and  $\oplus_{\delta \in S_r} V^{\delta+\gamma+\alpha}$ , we obtain the uniqueness of vertex operators up to the change of basis.

This completes the construction of VOA.

**Theorem 7.1** *Under the hypotheses II (1)  $\sim$  (8),  $V$  has a unique VOA structure.*

**Remark 10** *Except the last two assumptions, they are conditions on the codes. So if we are given a pair of codes  $D$  and  $S$  satisfying the conditions (1)  $\sim$  (5) and choose the desired  $M_D$ -modules  $V^\alpha$ , what we have to do is to check the associativity (7) and the commutativity (8). We should note that these conditions are possible to make sure among modules generated by three modules  $V^\alpha, V^\beta, V^\gamma$ . Sometimes, these information are possible to get from the known VOA. Namely, if there is a VOA  $V$  containing a set of mutually orthogonal rational conformal vectors  $\{e^i : i = 1, \dots, n\}$  with central charge  $\frac{1}{2}$  whose sum is the Virasoro element, then we obtain the associativity and the commutativity in the decomposition  $V = \oplus_{\chi \in \text{Irr}(P)} V_\chi$ .*

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